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THE MATHEMATICS TEACHER

Official Journal of the National Council
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[illegible] *Interests of mathematics in Elementary and Secondary Schools*

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THE MATHEMATICS TEACHER

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Edited by William David Reeve

An Analysis Showing Where Guidance Might Be Helpful in the Teaching of Mathematics

By ANN C. PETERS
New York, N. Y.

"High school students live abundantly; through guidance high school teachers should help them to live richly. Between the abundant life and the rich life there is all the difference in the world. The rich life has a sense of values, is discriminating, makes selections; the abundant life has but a single criterion of worth—satiation, sufficiency, fullness. The abundant life has no restraints, either intellectual or emotional; the rich is self-controlled and self-directed."

L. A. WILLIAMS: *Secondary Schools for American Youth*

ACCEPTING the thesis that through guidance children can be helped to live richly, we might well go further in stating that guidance and education are aspects of the same thing—the development of the whole individual. If guidance and education are the inseparable twins, then the task of directing personality, intellectual and social development falls heavily, indeed, upon the classroom teacher; for it is he who determines when the child and when the subject matter shall be taught. This, then, is the problem facing the classroom teacher of today, and the mathematics teacher in particular, as his complexities seem to have "varied directly as the increase of the high school population."

Before attempting proposals for guidance in the mathematics classes, the writer presupposes that an organized counseling system is operating in the high school. This department has a threefold responsibility:

1. Maintaining an accumulative record for each pupil
2. Diagnosing the results accumulated
3. Guiding the pupil's program in terms of the diagnosis.

Regarding the collected data, it is pertinent that by the time a student reaches the tenth grade the record should reflect the whole child for his entire school life. This is the complete accumulative record available at all times. It shows his strength and weakness in former grades, physical defects, remarks by former teachers, attitudes, interests, skills, attendance, and home life. Also in this record are the results of tests indicating intelligence, achievement, and aptitudes.

With the cumulative data in hand, the counselor diagnoses the student in terms of the results. He attempts in this case to place the child mathematically—the place from which the classroom teacher must begin if both he and the pupil are to be successful co-workers. If any doubt exists, the counselor may confer with the teacher of mathematics concerning the particular case.

With the diagnosis at hand, the counse-

lor and the pupil determine a program of study. This, indeed, is the crucial point and wise decisions, we hope, are made. If the pupil is beginning the ninth grade, he will often find several avenues open to him—algebra, general mathematics, or no mathematics whatsoever. Should his ability be average or better and the academic course is preferred, algebra is his solution. The same type of student, but undecided as to which course to pursue, should have the opportunity for exploration in a good general mathematics course and then reach a decision by the end of the term. The average and the superior pupils who do not elect to continue mathematics are of special concern to the writer. It is unfortunate that they remain "untouched" from the general exposure and it is at this point that counseling plays an important role. The lower than average, too, should be able to explore in general mathematics at their level of ability; for we cannot escape the fact that citizenship in a new world demands that all be trained at their level of ability to comprehend the basic mathematical concepts of our social and economic problems. Fortunate is he who is able to appreciate the truths of mathematics which help one somewhat to understand this civilization and the new world which is struggling in the atomic era to bring more of the good life to all peoples. As Wren[1] so aptly states, "Our youth must shape their programs to improve youth's possibilities for more successful and happier living in this era of great social change."

The mathematical courses having been determined, the scene shifts from the counselor to the classroom. Since mass instruction has its serious drawback in not being able to meet the needs of each pupil, the first item in classroom guidance is homogeneous grouping of abilities. Both the large and the small school might well operate on the three track basis—the superior, average, and slow groups. Criticism that this is unnatural, that it lacks competition, and that the slow group feels humiliated can all be overcome. We cannot afford not to train our potential leaders. Instead of competition within a group, encourage cooperation and stimulate self-competition. In the slow group the pupil can at last develop a feeling of success in being able to do the work. More than anything else, a child has the right to be happy in his work, and that cannot be true if he is working beyond or below his capacity. How he feels about what he learns cannot be overlooked.

In smaller systems the three levels are frequently within the one classroom where the instruction should be adapted to their needs and abilities. On this, or any other basis, the groups should not be labeled. Rather, "John's group" or "Jane's group" should be sufficient for identification. The instructor, beforehand, might state that since the class is large it will be better to work in groups where each one can do his best. Then, too, borderline students can be expected to shift between two groups. When need arises for a general committee for the room, the members should be chosen from all three groups. In assignments all meet the basic requirements, with added enrichment for the average, plus much enrichment for the superior as "their interests need not only to be guided but to be awakened and expanded through appropriate experiences."[2] This plan provides a study period for each group within the class period, thus eliminating all or almost all of the homework. This gives the instructor an opportunity to see the child at work and to set him straight in his mathematical difficulties at the time it needs to be done.

Regarding guidance via methodology, the old-fashioned "drilling in a vacuum" no longer exists. Rather, the guidance-minded instructor is concerned with the building of concepts and principles through *understanding* and learning in a variety of

[1] F. L. Wren, *Counseling in Mathematics*, unpublished paper.

[2] Butler and Wren, *The Teaching of Secondary Mathematics*, p. 70. McGraw-Hill Book Company, Inc., New York, 1941.

concrete situations and with a variety of presentations. Hence, drill becomes meaningful practice and skills are learned with attention.

Besides working at a child's level of capacity and building concepts and principles to promote guidance, one must also ask, "Are his interests being reached?" As education is life itself, his mathematical problems should be within the realm of the child's experience and presented in the light of their relation to everyday life. If a vocation has been determined, he should know its mathematical requirements and work an occasional problem from that field.[3]

In addition, the mathematics classroom itself should contribute to the guidance spirit. A cheerful room is conducive to study, and available material makes study during the period possible. In connection with the classroom there is a mathematics "lab," or the rear of the classroom serves this purpose. Here one finds a library with more than mathematical texts—there are histories of mathematics, popular editions of mathematics, biographies of mathematicians, books on mathematical recreation, vocational guides, etc. In short, this library vies with the library of the English classes. There are glassed cases for exhibits and geometric forms. There are large tables and chairs, cupboards stocked with every kind of mathematical instrument (home made and purchased) which, when used, bring "dead mathematics" to life. There is a file containing suggestions, illustrations, portraits of the masters, tests, enriched assignments, bulletin board materials gathered by both the pupils and instructor, and information concerning visual aids. This is the classroom where it is fun to work and which children are reluctant to leave.

The instructor, the key to the guidance program in mathematics, is as much a student of human nature as he is of his subject. He is sympathetic with the learner but not overly so. He listens to difficulties and aids the pupil indirectly to make decisions to help himself. He is confidential with personal information. Youth loves activity and learns by doing, and he is a wise teacher who capitalizes on this characteristic. He sees details and sequences which make up the entirety and teaches to bring out the whole pattern. He gives the children the pleasure of discovering relationships and teaches to make this possible. He believes in previewing a new unit and his reviews are new views. He correlates all branches of mathematics at every opportunity and with other subjects when possible. He insists on neatness and organization of work in notebooks. He teaches his pupils through mathematics to estimate, generalize, evaluate, and to note similarities between the old and the new.[4] Moreover, he is a teacher who understands youth, and they understand and have faith in him. He is inspiring and he has lived. He is alive and so are his pupils.

Hence, the counselor accumulates records, diagnoses the results, and with the aid of the pupil determines his mathematical course. The mathematics teacher who is concerned about guidance finds abilities, aptitudes, and interests as points of departure in aiding the child to equip himself for occupational and social adaptation. Working at his level of ability in a physically well-equipped classroom with an understanding instructor, the pupil has the opportunity to acquaint himself with the mathematical phases suited to his well being. In a subject field having the continuity of mathematics, or in any field which demands mathematics as foundational, the mathematics teacher's responsibility in the classroom cannot be underestimated in the total guidance program.

[3] National Council of Teachers of Mathematics, *Seventeenth Yearbook, A Source Book of Mathematical Applications*. Bureau of Publications, Teachers College, Columbia University, New York, 1942.

[4] Smith, Reeve, and Morss, *Teaching of Junior High School Mathematics:* Chapter III, Objectives to be Obtained. Ginn and Company, Boston, 1927.

Coordinating High School and College Mathematics*

By W. D. Reeve
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The problem of coordinating high school and college mathematics is one in which both high school teachers of mathematics and college teachers of the same subject should cooperate in solving. Failure to coordinate these separate fields in the past has led to a great deal of confusion and genuine loss both to the students involved and also to their teachers.

In the first place, the high schools themselves have not and still do not succeed in teaching their students as much as they are capable of learning if only they would wake up to their responsibilities in the matter. Two reasons may be given for our present failure to teach more mathematics to those who are capable of learning it. They are both related to the general problem of organization of content material.

1. Our traditional water-tight compartment method of teaching algebra, then plane geometry, then intermediate algebra, and so on, leads to a great deal of unnecessary repetition of subject matter throughout the secondary school period that results in the loss of a great deal of time and energy. We teach algebra in the ninth grade, then plane geometry is taught in the tenth grade. By that time the pupils have forgotten the algebra they knew so that when intermediate algebra (which should be a half year course as such) is presented usually in the eleventh grade, a large part of the time is spent if not wasted in reteaching ninth grade algebra. As a result it often happens that the entire eleventh year is spent on algebra when with better organization at least the ordinary course in trigonometry might have been presented. Then if the pupil goes on with mathematics he may be taught college algebra where again a great deal of time is spent, if not lost, reteaching elementary and intermediate algebra. And as is often the case with such practice, very little college algebra *per se* is presented.

What is even worse so far as the pupils and even their teachers are concerned is for the pupils to take the algebra, trigonometry or other really high school mathematics courses in the colleges and the universities, for one reason or another, and thus often miss analytic geometry and the calculus.

One excellent way to avoid all of this loss is to organize a course in general mathematics extending all through the high school so that those who can take it will have had a chance to study all of the mathematics, up to and including the fundamental ideas of the calculus, before they leave the high school. The college teachers of mathematics could then go on from there and devote their time to reorganizing what should really be mathematics on the college level and in that way make it possible for these boys and girls, whose life work depends upon a strong background of mathematics, to secure the necessary training without so much loss in time and energy.

I would like to point out here that it is most unfortunate that the idea that general mathematics is a course for dullards that keeps bobbing up in educational circles is not only unjustifiable, it is absolutely untrue. It is no doubt due to the fact that those who so consider general mathematics do not really know what it means, and that they have had little if any real experience in organizing and administering such a course.

If algebra were made the core in the ninth grade, geometry (plane and solid) in the tenth, algebra and trigonometry in

* Paper read at the Fifth Annual Meeting of the Mathematical Association of America, Metropolitan New York Section, at The Cooper Union on Saturday, May 4, 1946. Reprints may be had postpaid for 10¢ each.

the eleventh, analytics and the calculus in the twelfth and the necessary arithmetic taught in the course all along the line, it is demonstrable that not only much time could be saved, but the pupils concerned would have a much better background for their future work in mathematics and related fields.

The fact is that in a real important sense it takes a higher type of pupil to succeed in such a course than an inferior one.

2. Our failure to take account of individual differences in ability among high school pupils to say nothing of differences in interests and experiences, has resulted in great educational loss. The most retarded pupil in the secondary school today is the gifted pupil, the one with a scholarly mind. The secondary school machine is all geared up to turn out a mediocre product. This is obviously also true in the colleges. As Olsen[1] recently put it:

"An even more fundamental reason for insisting upon standards of some kind in every type of education is the value of self-discipline. Whatever may be said of the unwisdom of disciplinary measures thrust on a person by someone else, there can be no doubt that success in any field is achieved only by those who are able to determine their course and then hold themselves to it. Whether we interpret education as life itself or merely as preparation for life, it must, in some way, furnish this same basic life situation. Much harm is being done in our schools and colleges by lowering standards to such a point that the students find no stimulus in their work. We talk bravely about challenging our students, yet we set our objectives so low that there is no challenge for most of them. The solution to this difficulty would be to group students according to their interests and abilities, and then set up appropriate standards which would challenge each group. Modern psychology tells us that virtually none of us succeeds in developing all his potentialities to the highest possible degree. That is painfully evident in our schools today. Many students with genuine ability go through college with poor work habits, largely because they early found they could make satisfactory grades with very little effort. Because the work was planned for the poorer students, there was little incentive for those with ability to do more. Undeveloped potentialities are surely as great an evil as a frustrating experience of failure."

Again we must not forget that for some minds in these days of mass education the kind of mathematics that the gifted pupils need is not the same for the slower type of mind. We must remember that in a democracy we are under great responsibility not only to train leaders, but also to develop *intelligent* followers as far as possible in the schools. We should develop at least a two-track if not three-track system to meet individual needs.

Another difficulty that we face in trying to organize an adequate program of mathematics in the secondary school is the matter of reconciling points of view particularly among the general educationists and the teachers of mathematics in the schools and colleges.

President James Bryant Conant of Harvard University has recently given an excellent suggestion. In discussing the reasons why the lay critics of secondary education talk as they do, President Conant said:

"I am almost tempted to generalize that the more educated the person, the less his knowledge of secondary-school education. Certainly the lack of knowledge among the professors of arts and sciences in our colleges and universities is proverbial. And with lack of information goes lack of understanding and lack of sympathy. As a result, on more than one campus we have almost a state of civil war between those who profess a knowledge of education and those who profess a knowledge of subjects which constitute a

[1] Olson, Claire C., "And Gladly Would He Learn," *School and Society*, April 27, 1946, pages 289–291.

modern educational curriculum.

"This academic war has been in a sense inevitable, as I propose to show by a brief résumé of history, but to my mind an armistice has been for some years overdue. And it is for such an armistice that I should like to put in a good word this afternoon (and I might remark parenthetically that it takes two to make an armistice quite as much as to make a quarrel). My belief in the need for the cessation of hostilities comes not only from my general tendency to favor pacific methods of handling academic controversy, but also because I am really worried about the present lay reaction to the educational matters. I am distressed by both the vehemence and the ignorance with which views about education are expressed publicly and privately by many prominent people. Now we can hardly expect the public to have a very clear understanding about educational problems when education is a house warring against itself. Hence my plea this afternoon for a 'cease firing' order."[2]

This is the attitude that we have been taking for some time. When teachers of secondary mathematics, college mathematics, supervisors, administrators and general educationists sit down around the table to discuss what for all of them should be a common problem, then we can hope for some practical solution of what shall constitute general education in the post-war years.

We have a few outstanding cases[3] of men and women in the colleges and universities who have taken great interest in the work of the secondary schools and in organizations of teachers in secondary schools, but for the most part many of these people do not know what is going on in the schools and some of them seem to care less. And this in spite of the fact that some of the same pupils that are being trained in the mathematics of the secondary schools are the same pupils who will sit at the feet of the college instructors in mathematics in the colleges and universities in the days ahead. There is a great need for a better working understanding and cooperation between teachers of mathematics in the secondary schools and those of collegiate grade.

Let us now consider what should he done all along the line to improve the mathematical background of our people so that in peace time as well as in war time they can function as intelligent citizens whether or not a definite crisis exists.

In a recent editorial in *The National Mathematics Magazine*, Professor S. T. Sanders said:

"Many teachers are nervously concerned over what may be the post-war status of school mathematics. The enormous expansion of the technical applications of the sciences under pressure of war has brought about a world-wide strengthening of mathematics in the school curriculum. Can this current academic primacy of mathematics be made permanent? Such is the question raised by those keenly mindful of the scant attention paid to this subject by the less recent curriculum makers.

"A careful study of the matter should not discount the fact that in respect to mathematics, the war has served only to bring about greatly multiplied *uses* of mathematics, a large proportion of which were already in existence. For, even in pre-war times there had been for many years a steadily growing public emphasis upon *applied* mathematics, rather than upon the logical or cultural aspects of the science.

"In the light of this definite trend, a trend not rooted in any war, it could well be that the post-war school effort should first be directed to discovering the mathematical aids or needs of all the major peace-time industrial enterprises. Cooperative programs initiated between industry

[2] Conant, James Bryant, "A Truce Among Educators," *Teachers College Record*, December 1944, pp. 157–163.

[3] Witness the great interest of men like Professors Cairns, Hedrick, Kasner, Mitchell, Newsom, Slaught, and others.

and the schools would then have sounder foundations. Who shall say that the cultures of mathematics would be impaired by being stemmed in its utilities?"[4]

That great interest is being manifested in what place mathematics is to have in the schools after the war is evidenced by the many discussions that one hears, the various articles now appearing in current magazines and editorial comments like that above. Moreover, the National Council of Teachers of Mathematics has a *Commission on Post-War Plans* in mathematics the first report of which appeared in THE MATHEMATICS TEACHER for May 1944. A second and more inclusive report of progress of this commission appeared in the May 1945 issue of that journal.

The problem about which we are concerned here is that we do not have agreement in all quarters as to what should be done. We cannot take the space to survey all of the recent opinions and articles, but a few typical ones will show how the wind is blowing.

After discussing the high place which mathematics once held in the schools and the poor results in mathematical education shown by recent Army reports, Professor Harold L. Dorwart recently said:

"At this point, it may be asked why so many of our young people ceased to study mathematics some years ago. Many mathematics teachers say that it is all the fault of the educationists with their half-baked theories of the non-transfer of training and of the removal of everything difficult from education in order to prevent harmful personality development. The retort of the educationists is that mathematics teachers are just a lot of sadistic drill-masters who do more harm than good anyway. I will pass over this charge and countercharge in favor of what may, it is hoped, be a somewhat more helpful point of view.

"But, first, let us face the fact that there have been, and still are, both poor and poorly prepared teachers of mathematics, and that they have repelled many good students. Many high-school principals must be forced to give up the idea that anyone who possesses the credits in methods-of-education courses specified by law is thereby qualified to teach algebra or geometry. Also, college presidents and deans should investigate carefully the personalities of the instructors assigned to elementary courses. Even if the instructor is a recognized authority in his field, the conceited, arrogant, show-off type (fortunately few in number) should be used elsewhere than in elementary courses.

"I now propose the thesis that, once the requirements were withdrawn, students ceased to study mathematics principally because they did not recognize the fundamental rôle that it plays in modern civilization, or that by omitting the study of mathematics they were thereby imposing large restrictions on their future choice of profession or employment. In short, they did not recognize and usually have not been told that, in addition to serving, mathematics is a queen in her own right, a queen who will richly reward her followers, but only if they follow her diligently from their youth through a long period of time, even when the going is tough and when the path ahead is not always crystal-clear."[5]

In commenting on Professor Dorwart's article, the late Professor Bagley said:

"The leading article in last week's number, 'Mathematics—Queen and Handmaiden,' represents a type of discussion that is likely to become more and more important as the educational trends of the past few decades come increasingly to be scrutinized and questioned in the light of the educational weakness that the experiences of the war have revealed. Among these trends, probably none have been more debilitating than have the theories

[4] Sanders, S. T., "Post-War Planning in Mathematics," *The National Mathematics Magazine*, October 1943, p. 2.

[5] Dorwart, Harold L., "Mathematics—Queen and Handmaiden," *School and Society*, October 14, 1944, pp. 241–243.

and arguments used to discount and discredit the subjects that are, to use a favorite phrase of the present writer, 'exact and exacting, systematic and sequential'—more specifically, on the secondary level, mathematics and Latin.

"It is clear enough now that an educational development, fundamental in character and of far-reaching scope, lay back of the readiness with which American education accepted theories and postulates that served to rationalize this discreditment. The upward expansion of mass education, which has resulted in what is virtually a 'universal' high school, and which is now extending beyond the secondary level, would have been impossible without a relaxation of standards that are perhaps beyond the capacity and certainly alien to the tastes of a significant proportion of each generation of pupils and students.

"All this is water over the dam. The reinstitution of Latin and of mathematics beyond the simple arithmetical process as secondary-school requirements would doubtless be as unwise as it would be impossible. But it would be equally unwise, and it is not at all necessary, to continue allegiance to an educational theory that in effect encourages the following of the lines of least resistance by those who are competent to the more difficult types of learning.

"Expert educational guidance, coupled with such masterful teaching as Dr. Dorwart suggests, can do much to solve the problem. But there should be as well, we think, a systematic, even a militant, effort to enlighten both youth and the public generally as to the basic significance of the subjects that constituted the core of liberal secondary education over the long period during which the secondary school was a highly selective institution. It was no more an accident that this core comprised Latin and mathematics than it was an accident that the core of elementary education from the outset comprised the reading and writing for the mother tongue and the primary arts of number. (After all, civilization itself began with the invention of writing and the development of computation and measurement.) Liberal education on the secondary and higher levels has been based upon a refinement and expansion of the symbolism of conceptual thought. This has meant mathematics and, in English-speaking countries, Latin, which in a very real sense (as can be shown by abundant evidence), is the 'mother tongue of our mother tongue' in so far as the symbolism of advanced conceptual thought is concerned."[6]

Thus far the comments and articles referred to have been favorable to mathematics. However, we must present a typical point of view which raises questions as to whether mathematics is as important in the secondary school as some people think. Professor Frank N. Freeman, Dean of the School of Education at the University of California, recently said:

"One of the outcomes of the war, in the opinion of many officers of the Army and Navy and of many observant laymen, is the revelation of gross inadequacy in the teaching of mathematics. The experience of the armed forces and of industry is supposed to have shown that vastly larger numbers of students should study mathematics and that they should study more mathematics—up to and including trigonometry. We may look, therefore, for a campaign after the war, to require the study of mathematics through trigonometry by a large share of high-school students. If such a campaign is launched, it will grow out of an idea that is about as sound as the belief that the psychological tests given in World War I showed 40% of the male population to be morons.

"The reasons that no such conclusion follows from the experience of the Army and Navy, not to speak of industry, are, first, that the demands of the services in war time are no reflection of their demands

[6] Bagley, W. C., "The Present and Future of Latin and Mathematics in Secondary Education," *School and Society*, October 21, 1944, pp. 259–260.

in times of peace, and, second, that the number of men who need straight mathematics to perform duties required of them is much fewer than has been implied by the published statements. To say that the schools should, in peace time, give all the preparatory training that may be needed in time of war is like saying that industry must be kept in continual readiness to produce 10,000 planes a month. Again, the number of men who acquire techniques in the Army or Navy which are based on mathematics is vastly greater than the number who are required to understand the principles of mathematics and their application. The same is true of industry. The implied statement that all the men in the armed forces and industry who perform technical operations require a knowledge of higher mathematics for such performance is the wildest exaggeration. The war does not teach that mathematics through trigonometry is a practical necessity for a large proportion of men, let alone women. The question of the kind and amount of mathematics which is good for the average person and should be an element in general education still remains open."[7]

In the same article Dean Freeman presents a series of propositions which he thinks may be taken as a platform for the reorganization of mathematics in general education as follows:

1. Only that mathematics is important for general education which the individual will use.
2. The individual will use only those mathematical ideas and operations which he has learned by or in use.
3. The individual will actually use only those processes that he has mastered and made thoroughly familiar to himself.
4. Understanding is desirable, but it comes best through familiar use first, and formal explanation afterward.
5. Mathematics is, or contains, a form of language which formulates and defines the quantitative aspect of experience and, therefore, stimulates and largely creates quantitative ideas and forms of thought.
6. Mathematics may properly be thought of as a language—that is, as a particular set or particular sets of symbols which represent special aspects of reality.
7. To have meaning in the thinking of the child or of the ordinary person in general, the use of mathematical symbols and operations should be developed in intimate and continual association with the real world of experience.
8. The mathematics for the million is that which gives clearer and more effective ways of thinking about the real world of experience because it has been developed out of this world of experience.[8]

It is gratifying to know that the Mathematical Association of America and The National Council of Teachers of Mathematics are now cooperating in a plan which is intended to coordinate the work in Mathematical Education. The Association has appointed a Committee under the chairmanship of Professor C. V. Newsom and the Council one under Professor L. F. Wren. Professor Newsom's committee has already recommended that each Section of the Association give active attention to the following points; perhaps part of the time at each annual meeting could be devoted to a discussion of them. The Committee also recommended that each Section organize a committee to push the cause of mathematical education and to maintain contact with the central group.

Point 1: *The need for mathematics teachers is so great that departments of mathematics in institutions of higher learning should start an immediate campaign to recruit mathematics teachers.* It is undoubtedly true that the inadequate training in mathematics possessed by many of the students coming to our colleges and universities is due to the fact that we are

[7] Freeman, Frank N., "Teaching Mathematics for the Million," *California Journal of Secondary Education*, May 1944, pp. 246–254. See also Caswell, H. L., "Progressive Education Principles Used in the War Effort," *Teachers College Record*, March 1944, pp. 385–397.

[8] *Ibid.*, pp. 246–254.

producing an insufficient supply of mathematics teachers. Available data are quite conclusive upon this point. Moreover, the situation is steadily becoming more serious. In fact, a recent report by a Committee of the North Central Association of Colleges and Secondary Schools indicates that the number of new teachers available in September, 1945, was only 28% of a normal pre-war group. Moreover, in a survey of the situation in eleven representative states, it was found that in 1945 only 110 persons with an expressed interest in mathematics completed courses of study entitling them to standard certificates to teach in the secondary schools. Such figures indicate the critical nature of our problem inasmuch as the actual demand for new mathematics teachers was probably at least thirty times the number available.

The report of the Committee of the North Central Association also states that the number of emergency certificate holders has leaped "to sensational figures." Some of the emergency certificate holders have had little more than a year of college work. In fact, the report states that the lower requirements now in force have attracted large numbers of teachers "who have previously been unable or unwilling to make minimum preparation for teaching jobs." In Ohio, 140 temporary certificates to teach mathematics were issued in 1945 as compared to approximately twenty in a normal year; this seems to be typical of the situation in most states. It is literally true that a large proportion of high school teachers of mathematics are incompetent in both training and ability.

In addition to the critical situation now existing in the secondary schools, your attention is called to the tremendous demand which is developing for teachers in our junior colleges. Dr. Lawrence L. Bethel, President of the American Association of Junior Colleges, recently reported that the 600 existing junior colleges will grow to 1000 within a decade. At present, 275,000 students are enrolled in junior colleges. Expansion programs now under way will increase junior college facilities to handle 500,000 students. Within a decade it is expected that 800,000 students will be registered in junior colleges. Dr. George F. Zook, President of the American Council on Education, has proposed that each state provide for a system of junior colleges, each of which would be attached to a local cosmopolitan high school. Where will the junior colleges obtain mathematics teachers for this new program?

It is apparent to the members of the Coordinating Committee that departments of mathematics have a very definite responsibility to encourage many of their better students to consider teaching as a career. Any student who can make even an average grade in the calculus probably has a better background in mathematics than two-thirds of those now teaching high school mathematics.

It is undoubtedly true that the low salary scale generally associated with teaching has been a deterring factor to young people who are making plans for a career. It should be emphasized, however, that there is tremendous agitation at the present time to increase the salary scale for public school teachers; salaries have been rising during the past decade, and it is reasonable to assume that there will be considerable improvement during the next two or three years. In Ohio, the salary of the average high school teacher in city systems has risen from $1600 in 1934 to about $2500 in 1944. One prominent educator recently pointed out to the chairman of the Coordinating Committee that the salary figures usually quoted are not entirely fair to the profession. Many of the larger city school systems have salary scales which enable the members of their faculties to live upon an economic level at least comparable with that attained by college and university professors in our better institutions, In fact, many persons with training past the M.A degree and perhaps equivalent to the Ph.D degree can

actually anticipate a higher salary in public school work than they can possibly hope to obtain in college work.

Point 2. *College and university men must exert their influence for an improved secondary curriculum in mathematics, and probably should endorse the "double-track curriculum."* Mathematicians must accept a very definite responsibility toward public education by continually insisting that prospective college students have sufficient background in mathematics to be able to do satisfactory college work in the mathematical disciplines. In our zeal, however, it is essential that we remember that the public schools of the country are faced with some very difficult and important problems. The recent report of the Harvard Committee calls attention to the situation in a vivid fashion when it points out that in 1870 three-fourths of those who attended high school went on to college, so the high school's function was clear; it was simply to prepare for college. Now, three-fourths of the high school students do not expect to go to college. So we are faced with the fundamental question, "Can the interest of the three-fourths who go on to active life be reconciled with the equally just interest of the one-fourth who go on to further education?" To complicate the problem, "in many states nearly the whole population of high school age is now in high school, and the same may presently be true of most states. Thus, within a generation the problem of how best to meet this immense range of talent and need has grown up, like the fabled beanstalk, to overshadow virtually every other educational problem. It is in truth at the heart of any attempt to achieve education for democracy."

Educators familiar with the problems of secondary education seem to agree that some parts of the traditional college preparatory courses in mathematics require a higher level of intelligence then that possessed by a large proportion of our high school students. Moreover, it is doubtful that some of these courses really have value from any point of view for those students with limited ability. Consequently, there seems to be little hope that better prepared students will be coming to our colleges and universities until there is some segregation of high school students upon the basis of ability and future needs in mathematics. As college men, it is doubtful that we can tell the administrative officers of our high schools just where and how a division of the student body should be made. Nevertheless, we can insist that some special provision be made for those students who have superior abilities.

Our colleges and universities want their prospective students to study courses in high school that prepare them for college, but, as Professor MacDuffee has recently stated, "in advocating the double-track curriculum in the secondary schools, we must be careful not to call the express track the 'college preparatory' track. A superior student in high school has a right to a superior education even if he is financially unable to go to college. The fact that his financial situation is more subject to alteration than is his mental equipment should impel a teacher to be very cautious about advising a superior student to elect a vocational course. Many of life's little tragedies occur when students discover that their school work is of no use to them in preparation for their college work."

In addition to Professor MacDuffee's recent plea for increased attention to a double-track program in high school, the Post-War Planning Commission of the National Council of Teachers of Mathematics has been emphasizing a similar point of view. Also, the A.A.A.S. Cooperative Committee on Science Teaching is urging high schools to develop a program which will permit the introduction of an adequate mathematics curriculum for the able student who may go to college. The A.A.A.S. Committee suggests in the case of the small high school that a teacher might teach simultaneously two courses in

mathematics in the manner employed by every teacher of a one-room rural school. Thus in a single period for ninth and tenth grade pupils the teacher might teach algebra to one small group and general mathematics to the others. Of course, such courses as trigonometry and advanced algebra might be offered in alternate years. The further suggestion is made by the A.A.A.S. Committee that local boards of education might pay students' tuition for correspondence courses or for Saturday courses taught in larger schools in neighboring cities.

It appears to the members of the Coordinating Committee that the most concrete suggestion which has been made for a partial solution to our problem is the development of a double-track curriculum in all high schools. Then those students with limited mathematical ability and interests cannot hold back the competent students: moreover, teachers will be able to accomplish more with the poor students in a program of vocational mathematics when it is possible to work with a more homogeneous group.

Point 3. *Consideration should be given to the construction of a special curriculum in our colleges and universities for the training of prospective mathematics teachers.* A member of the A.A.A.S. Cooperative Committee on Science Teaching recently wrote, "The committee is convinced that progress will be made in the preparation of science teachers only if the science departments consider that this preparation is one of their major functions and treat it as such. There should be a section in the college catalogue describing the program for the preparation of science and mathematics teachers. This program should be worked out cooperatively by the science departments and the department of education. Such collaboration will result in a better solution of the problems of student teaching and the courses in methods of teaching, which the committee recognizes as an essential part of the preparation of a teacher."

The criticism is constantly made by secondary educators that mathematics and science teachers seem to have an inadequate appreciation of the historical, social, and philosophical backgrounds of their subjects. Moreover, it is surprising how many teachers are attempting to teach high school mathematics without any background whatsoever in the applications of their subject. As a partial remedy for whatever justification there is in comments such as these, it seems imperative to the members of the Coordinating Committee that in addition to the usual basic undergraduate courses in mathematics, a prospective teacher of mathematics should have a course in the history of the subject and a course in the foundations of mathematics. Also, a prospective high school teacher can hardly afford to slight the study of basic courses in physical science and, to a certain extent, in the social sciences.

Your attention is also called to the fact that high school teachers with an A.B. degree must usually start their teaching careers in the small high schools. Half of the high schools in the country have five teachers or less. Three fourths of them have ten teachers or less. Yet, the small school must offer courses in at least twelve subjects, and many of them offer fifteen to twenty subjects. Consequently, it is inevitable that beginning teachers of mathematics must also be prepared to teach some other subjects. A survey made in the state of Indiana for the year 1940–41 revealed that 272 mathematics teachers also teach some kind of science. The next most popular combination is mathematics and physical education. The comment is frequently made that the high school coach is also the mathematics teacher. The study in the state of Indiana indicates that there is some truth in the assertion, but the mathematics-physical education combination is not nearly so common as the mathematics-science combination. It is our belief that the survey in Indiana is rather typical of the situation generally.

The A.A.A.S. Cooperative Committee is strongly urging, therefore, that prospective teachers of high school science and mathematics should study as much as sixty semester hours in the combined fields. A total of sixty semester hours would permit a division among three subjects such that there could be twenty-four hours in a major subject, such as mathematics, and eighteen hours in each of two others. The A.A.A.S. Committee recognizes that such a program does not provide adequate specialization for the teacher going into larger school systems. The conclusion at this point simply is that more and more capable students interested in teaching should be encouraged to spend a five-year period in preparation. Of course, most of the larger secondary schools of the country already require the M.A. degree or its equivalent for those teachers who hold permanent positions.

It seems obvious to the members of the Coordinating Committee that each department of mathematics in an institution of higher learning might well take stock of its program for the training of teachers. The time has come for a thorough evaluation of our relationship to the public schools of this country.

What groups of American citizens them are likely to need further training in mathematics in the days ahead and what type of organizations of subject matter and methods of instruction are likely to prove most efficient?

The purpose of a course in mathematics in the secondary school is to meet the needs of four groups of pupils:

1. Those who intend to go on to colleges and technical schools
2. Those who are going to specialize in commercial work or vocational pursuits that require mathematics, especially algebra
3. Those who intend to major in science
4. Those who elect mathematics because they like it.

There is no question about the need for mathematical training for the pupils in the first group above who are going to be engineers, statisticians, actuarial workers, certified public accountants and the like. The main problem here is one of guidance. If we can be certain that pupils are on the right track, then we can and should clear the way so that such pupils can be given not only four years of mathematics but mathematics of a very high order. To permit such pupils to take less mathematics and waste time in their course by going at the rate of less superior pupils will be tragic for them and the country as well. The same line of thought follows for those pupils who plan to go to college where they will specialize in mathematics and then teach the subject, carry on research, or be applied mathematicians of one type or another.

A second group of skilled workers in business and industry will need to know certain mathematical facts and be able to use certain mathematical skills in the new era that will open up after the war. Their work is extremely important, but they will not have to be as highly trained as those in the first group mentioned above. The possible future needs of such a group needs to be studied carefully and a suitable course of study worked out to meet such needs.

A third group consists of those who intend to specialize in science. We ought to be able to assume that for them mathematical training of a high order would be basic if it were not for the fact that it does not work out that way. A careful study of the needs of this group should be made and the necessary content material organized for teaching purposes.

The last and by far the largest group of our citizens will be those who take mathematics because they are made to realize that without a reasonable understanding of the fundamental ideas they will not be competent to carry on as they should. The skills and habits of quantitative thinking must be acquired by such pupils if they are to meet the reasonable demands placed upon them.

In the new era just ahead many of our

pupils will need to know how to solve the problems of air and marine navigation. This will necessitate an understanding of the facts and processes of algebra, geometry and at least numerical trigonometry to an extent that will make the offering of at least a two-year course in mathematics starting with the ninth grade.

In the fields of business and industry where the war has created new problems, different emphases may have to be made particularly where some of these problems of war come to be also the problems of the peace days ahead.

Let us also hope that more and better prepared teachers will be available and at salaries that will enable them to have a decent living. No matter what happens, however, we should not expect miracles and it is clear to those who understand the present situation that in many places better teachers, higher salaries, and more money for better books, equipment and the like may not be forthcoming.

In any case teachers who remain in the classroom because they want to do so in spite of handicaps must somehow learn to teach in better and more interesting ways many of the things that are now in the course of study. This need not be discouraging, for this can be done, if and only if, teachers are inspired to carry on the work as outlined by the commissions which have tried so hard to help.

California Mathematics Council

TENTATIVE PROGRAM

STANFORD CONFERENCE, FRIDAY AND SATURDAY, DECEMBER 27 AND 28, 1946

Theme: THE MATHEMATICS CURRICULUM

Friday Morning and Afternoon

A.M. Symposium with topics chosen from the following:
- Developing the Mathematics Curriculum
- The Elementary Curriculum in Mathematics
- Guidance in Mathematics
- Individual Differences in Mathematics
- Mathematics in General Education

P.M. Section meetings with group discussions
- Elementary section
- Junior High section
- Senior High and Junior College section

Theme: CLASSROOM TECHNIQUES IN ARITHMETIC

Saturday Morning

Demonstrations and discussions on such topics as:
- Development of number concepts and reading in mathematics through experiences in the elementary grades
- Use of experiences in developing concepts at the junior high school
- Reading in mathematics in the secondary school
- Remedial arithmetic in the high school

Theme: EVALUATION

Saturday Afternoon

Symposium and discussions on such topics as:
- Available tests and measurements
- Diagnostic program in the mathematics classroom
- Anecdotal records in teaching mathematics
- Evaluation of the mathematics curriculum

Reports and plans of Council committees will be discussed each day at 8:30 A.M. Recent films in mathematics will be shown each day at 4:30 P.M. Hotel reservations should be made immediately at The President or The Cardinal in Palo Alto. Write Miss Harriette Burr at San Jose High School if you are not successful in securing a reservation.

Control of Fundamental Mathematical Skills and Concepts by High School Students*

By Merle M. Ohlsen
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Statement of the Problem

The purpose of this study was to answer the following questions concerning the mathematical achievement of the students in grades ten, eleven, and twelve in forty-three selected Iowa high schools: (1) What degree of mastery of the mathematical skills and concepts described as essential for the ordinary citizen in the Final Report of the Joint Commission of The Mathematics Association of America and The National Council of Teachers of Mathematics[1] is attained by high school students? This question involved the dual problem of determining the degree of mastery for each defined concept and skill as well as the degree of mastery of the composite of these concepts and skills. (2) What common errors do students make in applying these concepts and skills? (3) Are there significant differences between grade levels in the degree of mastery of this basic mathematics?

Procedure

In discussing the essential mathematics for the ordinary life experiences, the Joint Commission[2] classified basic mathematics into four categories: arithmetic, graphical representation, algebra, and informal geometry. An outline of these concepts and skills facilitated greater efficiency in constructing the test. The mathematical needs of the ordinary citizens were formulated in terms of the attainments which the Joint Commmission[3] considered to be normal mathematical equipment of the American public who had satisfactorily completed the work of the sixth grade and the essential mathematics for the ordinary citizen, which they discussed in Appendix I.[4] From their discussion of this essential mathematics, the writer defined the following twenty-nine concepts and skills.

1. A familiarity with the basic concepts, processes, and vocabulary of arithmetic.
2. Understanding of the significance of the different positions that a given digit may occupy in a number, including the case of a decimal fraction. This covers the range from very small numbers to numbers as large as millions.
3. A mastery of the basic number combinations in addition, subtraction, multiplication, and division.
4. Reasonable skill in computing with integers, common fractions, and decimal fractions.
5. An acquaintance with the principal units of measurement and their use in every-day situations.
6. The ability to solve simple problems involving computation and units of measurement.
7. The ability to recognize, to name, and to sketch such common geometric figures as the rectangle, the square, the circle, the triangle, the rectangular solid, the sphere, the cylinder, and the cube.
8. The habit of, and therefore skill in, estimating and checking results.
9. Application of the fundamental arithmetic processes in the activities of ordinary life such as budgeting income, keeping accounts, checking

* Approved by Associate Professor L. A. Van Dyke, Chairman of Sponsoring Committee, College of Education, State University of Iowa. An abstract of a dissertation submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in the Department of Education, in the Graduate College of the State University of Iowa, January, 1946.

[1] The National Council of Teachers of Mathematics, Fifteenth Yearbook, *The Place of Mathematics in Secondary Education*, The Final Report of the Joint Commission of The Mathematical Association of America and The National Council of Teachers of Mathematics, 1940, pp. 54–65, 207–211.

[2] Fifteenth Yearbook, *ibid.*

[3] Fifteenth Yearbook, *ibid.*

[4] Fifteenth Yearbook, *ibid.*

bills, buying insurance, handling funds, wages, and social situations.
10. Understanding of the meaning and the use of per cent in problems dealing with installment buying, depreciation on the home or car, interest on loans and investments, taxes, wages and commissions, determining prices and profits, discount, and social situations.
11. Understanding of statistical data involving averages, including a knowledge of the method of calculating an arithmetic mean.
12. Understanding of and the ability to use data involving ratio and proportion.
13. Ability to interpret line graphs, bar graphs, and circle graphs.
14. Ability to interpret scale drawings.
15. Understanding of positive and negative numbers and their application in such situations as describing opposites, like assets and liabilities, and temperatures above and below zero.
16. Efficiency in the use of the formula.
17. Understanding of the dependency of one quantity upon another and of the relation of variables in a simple numerical table.
18. Competency in employing the algebraic equation as a general way of finding unknown quantities. This implies
    a. the ability to solve a simple equation
    b. the ability to write an equation which expresses the relationships existing in a verbal problem.
19. Understanding of ordinary rectangular coordinates.
20. Competency in determining distances and angles by the use of similar triangles and trigonometric ratios (especially the sine, cosine, and tangent; with emphasis on application).
21. Understanding of the use of the exponent and its application.
22. Understanding of the meaning and use of logarithms.
23. Familiarity with and facility in handling the application of permutations and combinations.
24. The ability to recognize at least the elementary basic terms of geometry.
25. Knowledge of the sum of the angles of the triangle.
26. Understanding of the rule of Pythagoras and ability to apply it in solving problems.
27. Ability to use geometric relationships involving mensurational calculations. This includes the recollection of the common formulas involved.
28. Proficiency in making direct measurement of lengths and angles with the ruler and protractor.
29. Ability to use the compass in making common constructions.

Several hundred multiple choice items were constructed to measure achievement in the defined skills and concepts. Two or more of these items were selected to measure achievement in each concept and skill. Several skills and concepts had to be broken down into sub-topics; therefore 107 test items were used. No student actually took a single test of 107 test items because these items were divided into five thirty minute tests. The precautions taken in distributing the tests insured random selection of the individuals who took a given test in the series of five. The five tests were submitted to several faculty members of the Mathematics Department and the College of Education to eliminate ambiguities, irrelevant clues, and technical errors. Moreover, these same individuals evaluated the ability of each test item to measure the concept or skill it was supposed to measure. After the items were carefully checked for clarity and validity, they were re-distributed to assure an equal time requirement for each test. Then the tests were administered to all of the students in grades ten, eleven, and twelve in one high school. Immediately following the administration of the tests the students were given an opportunity to point out problems which were not entirely clear to them. In addition to checking the clarity of the problems to students for whom the tests were designed, this try-out provided an opportunity for re-checking the time needed to complete each test. After making the necessary revisions the tests were printed and administered in the participating high schools during the week of February 12–16, 1945.

The students enrolled in grades ten, eleven, and twelve of forty-three Iowa

high schools participated in this study. In attempting to select schools representative of Iowa high schools two factors were considered: size classification of the school and the geographical location. The classes of high schools were defined as they were in the 1944 State University of Iowa Fall Testing Program. In this program the high schools were classified according to the enrollment in grades nine through twelve as follows: Class D—(65 and below), Class C—(66–125), Class B—(126–400), and Class A— (401 and above). This sample is composed of approximately five per cent of the schools in each of these enrollment classes in Iowa. Two thousand, eight hundred ninety-three students participated in this study. Approximately six hundred students took each test item. The individual who took a particular test in the series was selected at random from all the individuals in his grade within his school.

The per cent of the students selecting the correct response was taken as the index of achievement for each test item. Since there were two or more items covering each concept or skill, and since almost exactly the same number of students reacted to each of the 107 test items, the arithmetic mean of the indices of the items involved gave a good estimate of the achievement on each concept and skill, and also on the test as a whole. If a student's work showed that he understood the concept, but that he merely made a computational error, he was given credit for proficiency on the item in determining the degree of mastery for the concept if computation was not a basic part of the concept.

The common errors were determined by studying the selection of responses and the students' work. When the student selected any one of the first four responses the writer assumed that the student made the predetermined error unless his work definitely showed that some other error was made. Other common errors were discovered through a study of the answers given under "none of these." The fact that students were required to show all of their calculations in the space provided made possible this analysis of how they obtained their answers.

Since research evidence indicated that there was a positive relationship between achievement in mathematics and the number of mathematics courses taken in high school this factor was held constant in determining whether or not grade differences were significant. The writer selected all the students in grade 10, grade 11, and grade 12 who had completed one year of algebra and one year of plane geometry, or were in the process of taking plane geometry, but who had taken no other high school courses in mathematics. The per cent score on each item for each of these three special grade groups was substituted in the common formula for computing standard error of percentages. Then the standard errors of percentages were used by pairs in the common formula for determining the standard errors of differences. A grade difference was considered to be significant if the null hypothesis could be rejected at the one per cent level.

Summary

The performance of these high school students on test items evaluating proficiency in the basic concepts and skills essential for the ordinary citizen is summarized in Table I. Inasmuch as there are two or more items covering each concept or skill, the indices in Table I are the arithmetic means of the percentages for the individual items under each concept or skill. The performance of seven groups of students is described in Table I. The special groups (10s, 11s, 12s) had completed both first year algebra and plane geometry, or were in the process of taking plane geometry, but who had had no other mathematics courses in high school. The three grade groups (10, 11, 12) included all the students who took the individual test items. The performance for all of the students is summarized in the last column of Table I. The arithmetic mean of the

Table I

*Indices of Mastery for These Concepts and Skills**

Concept or Skill	10s	11s	12s	10	11	12	Total
1. Arithmetical Vocabulary	51.7%	47.6%	50.6%	50.2%	51.8%	57.5%	53.0%
2. Meaning of Numbers	64.6	67.7	69.2	64.7	68.3	70.8	67.8
3. Number Combinations	89.8	89.4	89.4	88.4	89.8	91.8	89.9
4. Fundamental Processes	72.6	75.1	66.5	71.5	73.9	75.5	73.7
5. Units of Measurement	46.2	47.6	53.0	44.1	48.2	57.7	49.8
6. Applications of Units (5)	61.6	64.6	61.5	65.2	64.9	63.4	62.9
7. Geometric Figures	60.3	66.7	54.3	53.9	59.8	58.2	57.4
8. Estimate-Check Results	62.4	67.3	60.8	61.3	65.1	67.4	64.5
9. Applications of Processes	59.0	57.2	59.1	58.0	59.2	64.6	60.4
10. Applications of Per Cent	28.4	30.1	28.3	26.7	28.2	34.1	29.5
11. Averages	87.2	89.0	90.4	85.9	89.3	91.4	88.9
12. Ratio and Proportion	42.6	44.7	51.5	43.9	45.8	56.5	48.5
13. Graphs	62.7	63.9	75.3	62.0	65.2	72.2	66.1
14. Scale Drawings	52.2	48.0	55.8	49.9	52.0	61.4	54.2
15. Signed Numbers	37.2	35.6	28.9	32.0	32.2	34.8	32.9
16. Use of Formula	42.1	42.3	42.0	38.5	41.8	50.3	43.4
17. Dependency	13.8	13.7	19.0	13.7	16.3	26.3	18.5
18. Algebraic Equations	34.8	30.8	26.9	34.0	33.5	36.6	34.6
19. Coordinate System	41.5	37.0	28.7	36.7	38.6	41.0	38.8
20. Indirect Measurement	21.3	27.2	30.3	21.7	24.1	32.1	25.7
21. Exponent	10.0	16.4	12.7	13.4	17.3	22.0	17.4
22. Logarithms	5.8	1.7	4.1	6.0	3.7	6.2	5.3
23. Permutations-Combinations	24.4	28.6	34.1	22.8	25.8	34.8	27.4
24. Geometric Vocabulary	64.8	65.5	51.3	50.0	53.8	53.6	52.4
25. Sum of Angles of Triangle	92.7	78.7	72.9	73.8	73.8	73.6	73.7
26. Rule of Pythagoras	8.7	11.1	10.0	7.8	9.7	20.0	12.2
27. Mensurational Problems	47.3	45.0	44.7	45.0	43.9	52.7	47.1
28. Direct Measurement	57.8	51.8	55.2	49.8	52.1	59.5	53.6
29. Common Constructions	71.7	68.3	63.5	51.7	55.8	59.9	55.7

* Approximately 600 students took each test item. The breakdown by grades is approximately 10s—90, 11s—60, 12s—40, 10—215, 11—200, 12—180.

five items dealing with mastery of common number combinations is 89.4%. This is the second highest general index for the special twelfth grade group. According to Table I the indices of proficiency for the totals of the three grades on these concepts and skills varied from 5.3% (number 22—Understanding and Use of Logarithms) to 89.9% (number 3—Use of Basic Number Combinations). Of the four concepts and skills with the highest indices of mastery, three were associated with arithmetic (3, 4, 11) and one with geometry (25). Three of the four concepts with lowest indices of proficiency were associated with algebra (17, 21, 22) while the other represented proficiency in a geometric concept (26).

Arithmetic

The arithmetic mean of the indices for all the items which were designed to measure proficiency in arithmetical skills was 75.0%. When the index of proficiency was determined on the basis of item indices under arithmetical concepts, it was found to be 49.2%. The performance on the problems which involved percentage was considerably lower than on those dealing with any other arithmetic concepts. The students did better on the arithmetic skills than on the arithmetic concepts. These high school students exhibited an average proficiency of 57.8% on all the arithmetic items in the tests.

The common errors which were made in solving the arithmetic problems fall into six major categories: (1) These high school students confused related mathematical terms; in particular they confused square with square root, sphere with circle, and compute with estimate. They were unable to distinguish between terms in the metric system, and also between various business terms. (2) Either inefficiency or carelessness in reading the problem frequently

resulted in failure to select the correct data or in missing the objective of the problem. (3) Lack of understanding of the significance of place value in numbers, errors in placing the decimal point, and incorrect use of the metric system all suggest inadequate knowledge of the decimal system. (4) These students were deficient in the skill of rounding numbers off to the defined degree of accuracy. (5) Miscellaneous mistakes in computation, such as errors in number combinations (particularly those involving zero), carrying and borrowing, and reducing improper fractions to mixed numbers were made. (6) Three important factors were largely responsible for the low performance on the percentage problems: (a) inadequate understanding of the meaning of per cent, (b) selecting the wrong number as the base, and (c) incorrect interpretation of the time factor in interest problems. Grade differences were not significant for those individuals who had approximately the same mathematical background in high school.

Graphic Representation

These students showed considerably higher proficiency in interpreting graphs than they did in interpreting scale drawings. The common practice of using graphs to present statistical data in newspapers and magazines might account for this difference. Computation of the arithmetic mean of the item indices in this subject category resulted in the composite index of 60.2%. Grade differences were not significant.

In interpreting the facts associated with a circle graph, the students confused "per cent" and "number of degrees" and had difficulty in selecting the essential data for the defined solution. A few students omitted the first and last pairs of coordinates in interpreting the line graphs and bar graphs. Determining the scale from the given facts proved to be more difficult than using the scale once it was determined.

Algebra

About 60% of these students could solve a first degree equation without fractional coefficients. In contrast, only one-fourth of the same group were able to solve first degree equations with fractional coefficients. The students' work indicated that they were about as proficient in writing an algebraic equation from facts given in the problem and in solving a second degree equation as they were in solving a first degree equation with fractional coefficients. The index for this skill was 34.6%.

In general, it can be said that students know very little about the fractional exponent, the zero exponent, and logarithms. Their opportunity to use permutations and combinations was limited to situations where it was practical to write out the possibilities and hence results may have been spuriously high. The index of mastery for these concepts is 27.2%. In Table I it can be seen that there were seven concepts and skills which had indices of less than 30%; five of these seven indices are associated with concepts and skills classified under algebra. For all the items classified under algebra the index of proficiency is 29.2%.

The six most common errors were associated with a specific concept or skill: (1) collecting and transposing terms in solving equations, (2) selecting data and stating them accurately in terms of the relationships and objective of the problem, (3) interpreting signed numbers, (4) differentiating between the trigonometric ratios, (5) misinterpreting the meaning of the exponent, and (6) confusing the relationship of corresponding sides in similar figures.

Informal Geometry

Fifty-four and seven-tenths per cent was the index of mastery on the items covering these geometric skills. This group was about as proficient in making direct measurements as it was in making common constructions. The average index for

TABLE II

Summary of Performance by Subject Categories

Categories	Grade Groups						
	10s	11s	12s	10	11	12	Total
Arithmetic	56.3%	59.8%	56.1%	55.0%	57.3%	61.4%	57.8%
Graphic Representation	57.5	56.0	65.6	56.0	58.6	66.8	60.2
Algebra	28.1	27.4	27.6	26.7	28.7	33.3	29.2
Informal Geometry	57.4	54.7	49.9	47.2	49.0	53.7	49.8

the items which measured an understanding of these geometric concepts is 48.0%. This index is slightly lower than the one for geometric skills. In this group of concepts only one had an index below 30%. This index is associated with the use of the rule of Pythagoras. For all items designed to measure proficiency in geometric concepts and skills the index was 49.8%.

The most common errors can be classified as follows: (1) inadequate knowledge of geometric relationships, such as confusing supplementary angles with complementary angles, acute angles with obtuse angles, diameter with secant, and congruent with similar; (2) lack of knowledge of conditions for congruency; (3) inability to select the data necessary for the correct solution; (4) inefficiency in using units of measurement in mensurational problems; and (5) confusing the steps in similar constructions.

The achievement of group 10s was significantly better than that of group 12s on Concepts Number 24 and Number 25. No grade differences were significant at the one per cent level on any of the other geometric concepts and skills.

Table II summarizes the performance in these categories by grade groups.

In Table II it can be seen that every grade group except 11s exhibited its highest performance on graphic representation. It is important to notice that the indices for algebra are only about half as large as those in the other mathematical areas.

The general picture of performance is summarized in Table III. The first column in this table represents the arithmetic mean of the per cents of *correct* responses to the test items. The second column in Table III represents the arithmetic mean of the indices of proficiency for the individual test items. When the ability to compute was not the primary purpose of the item, a student was given credit for proficiency on the item if he demonstrated an understanding of the correct method. The differences in the two columns in Table III can be explained in terms of students' computational errors.

TABLE III

Mean Achievement by Grade Groups

Grade Groups	Mean Per Cent of Correct Responses	Mean Proficiency on *Individual* Items
10s	46.2%	48.5%
11s	45.4	49.3
12s	44.0	47.2
10	42.6	45.5
11	44.7	47.5
12	49.2	52.2
Total	45.6	48.2

According to Table III the index of proficiency on *all* the *items* for the total sample of students is 48.2%.

The data in Table III indicates that there was a slight decline in proficiency from group 10s to group 12s. Two factors probably account for this decline: (1) the fact that these seniors had had no mathematics since they completed geometry, and (2) the fact that the better mathematics students may have elected to take more mathematics. On the other hand, it may be noted in Table III that the whole sample of students in the last three years of high school showed a tendency to in-

crease in proficiency from grade to grade. However, none of these grade differences is significant at the one per cent level. The tabulation which follows shows the preparation of these students in formal courses in mathematics.

Mathematics Background	10	11	12
Only first year algebra and plane geometry (group s)	43.2%	32.3%	20.3%
Less mathematics (than group s)	46.2	34.3	34.5
Only general mathematics and first year algebra	10.6	9.0	3.9
More mathematics (than group s)	0.00	24.3	41.3

This general tendency for gain is probably associated with the large number of the high school students taking mathematics courses beyond plane geometry.

From the data presented in this study the principal findings are as follows:

1. The test results indicate that the high school students tested did not demonstrate a high degree of proficiency on many of the mathematical concepts and skills defined by the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics.[5] On the average there were 45.6% who obtained the *correct answer;* 48.2% knew how to get the correct answer. The difference is due to the students' computational errors.
2. Classification of all of these concepts and skills into four subject categories and computation of the arithmetic mean for the item indices in these categories resulted in the following indices: (1) arithmetic, 57.8%; (2) graphic representation, 60.2%; (3) algebra, 29.2% and (4) informal geometry, 49.8%. The general proficiency index was 48.2%.
3. An analysis of the errors made by the students indicated that the most common errors occurred as follows: (1) through a lack of understanding of a correct method for solving the problem, (2) in confusing related mathematical terms, (3) in selecting the incorrect data for the specified solution, and (4) in errors in computation.
4. For the students who had had first year algebra and plane geometry, but but no other mathematics courses in high school, the proficiency tended to decrease between grades ten and twelve. However, the differences in achievement between grade levels were not significant.
5. For all of the students in grades ten, eleven, and twelve there was a tendency for proficiency to improve year by year.

[5] Fifteenth Yearbook, *ibid.*

Conclusions

The general index of 48.2% indicates that the present high school mathematics program is not efficient in preparing youth for life as an ordinary citizen. These data reveal a need for evaluation and revision of the high school mathematics program as well as a need for critical analysis of the method of teaching mathematics. The high schools must assume the responsibility for student mastery of the fundamentals in mathematics.

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Linkages as Visual Aids*

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Introduction

Although many fundamental concepts of mathematics may be demonstrated by the use of linkages, very few teachers have used them regularly in their classes. The purpose of this paper is to show some of the ways in which linkages may be used and to consider some of the concepts which the pupils may obtain in this way. Two principal topics are considered: (a) Linkages demonstrating fundamental concepts; (b) Linkages to arouse interest. The linkages under topic (b) may also be used to demonstrate fundamental concepts in advanced secondary school classes or in college.

A linkage is defined in the *Encyclopedia Britannica* as a system of bars connected by pin joints to allow deformability without sliding motion. Linkages may be made of strips of wood, cardboard and brass fasteners, galvanized iron and rivets or eyelets, or similar materials. For most classroom uses cardboard is satisfactory, although metal strips are more rigid and best if one has the equipment for utilizing them. I prefer strips of tinplate or galvanized iron and eyelets such as the surveyors use in mending their steel tapes.

At a meeting of the mathematics teachers of Rhode Island in 1940 a professor of engineering told us that his most difficult problem was to train his students to think of moving lines, of quadrilaterals in various positions, instead of static figures on the blackboard. These instruments supply one approach to his problem.

Historically, one of the first linkages was devised by James Watt, the inventor of the steam engine, in an attempt to convert straight line motion into circular motion. Although his linkage gave only approximate motions, it served a practical purpose and greatly facilitated the development of the steam engine. He wrote to his son, " . . . I am more proud of the parallel motion than of any other mechanical invention I have ever made."[1] Another highlight in the development of linkages occurred when a French army officer, Peaucellier, discovered a method of drawing a straight line—a theoretically straight line.

Before mentioning linkages which draw straight lines and other figures, let us consider a few very simple devices which have proved useful in the classroom.

Linkages Demonstrating Fundamental Concepts

The basic elements of a linkage are a bar and a pin joint. Although the bars used in the following linkages appear to have straight edges, this is not necessary. As one may readily see, the two bars at the left of Plate 1 will serve equally well for drawing circles. Similarly in all the linkages for performing various constructions the one property of the bars used is that the distance between any two points (usually indicated by eyelets) remains constant. However, in order to improve the appearance of the linkages, bars similar to the one on the right are used (Plates 9–13).

If we consider the upper eyelet of either bar at the left of Plate 1 as the center and the distance between the eyelets as the radius, it is clear that the locus of the corresponding lower eyelet is a circle. This demonstrates to the pupil that a circle is the locus of points equidistant from a fixed point.

* Presented at "The Institute for Teachers of Elementary and Secondary Mathematics," Duke University, Durham, North Carolina, June 17, 1942.

[1] James P. Muirhead, *The Origin and Progress of the Mechanical Inventions of James Watt*, Vol. III. London: John Murray, 1854, p. 89.

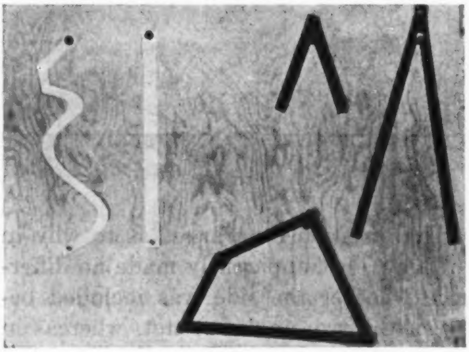

PLATE 1

There are many ways of illustrating angles to the pupils. Some teachers struggle with two yardsticks; others use blackboard compasses for angles less than 180°. Recently I have used two strips of cardboard joined by brass fasteners (Plate 1, upper right). This extremely simple linkage clearly demonstrates an angle as an amount of rotation showing the vertex, initial and terminal sides. It can be used to illustrate various angles and is especially good for angles greater than 180°. The straight angle is demonstrated as half of a revolution and the vertex is clearly indicated so that there is no question about its being a straight line. Another advantage of this linkage is in distinguishing between such angles as 45°, 315°, and 405° not in terms of the space between the bars but in terms of the amount of rotation which has taken place, as must be done in mechanics. Why not let each pupil have such a linkage or make one, manipulate it, and really understand what is meant by an angle?

After the pupil has learned that a triangle is the figure formed by three straight lines which do not all pass through the same point and no two of which are parallel—actually Plate 2, upper left but usually indicated as in Plate 2, upper right—he is shown some of the various types of triangles. Using the linkages in Plate 2 one can illustrate equilateral, isosceles, scalene, right scalene, right isoscles, 3-4-5 right, acute, and obtuse triangles.

Next we consider relationships between triangles.

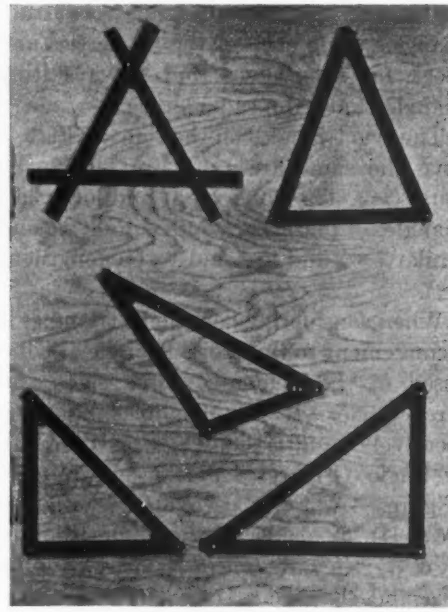

PLATE 2

The three triangles at the top of Plate 3 have the same shape and are said to be similar ($\sim$). The two smaller ones may also be hung inside the larger to show that their shapes are the same. The pupils may discover the properties of similar triangles by comparing corresponding

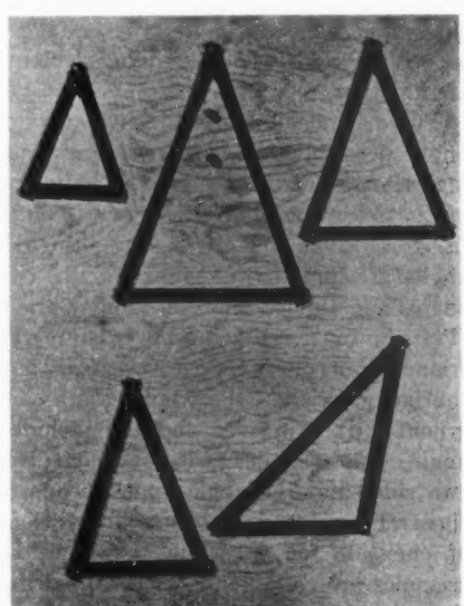

PLATE 3

angles and sides. Then they may construct triangles using each of the properties and check for themselves that the resulting triangles have the same shape.

The two triangles on the right of Plate 3 have the same area and are said to be equal (=). The upper right and lower left triangles of Plate 3 are both equal (=) and similar (∼) and are said to be congruent (≅).

When considering congruent triangles it is convenient to think of a triangle as composed of six elements—three sides and three angles. If two triangles have respectively equal a side (Figures 1A & B), an angle (Figures 1A & C), two sides (Figures 1A & B), two angles (Figures 1A & D), or a side and an angle (Figures 1A & C); they are not necessarily equal in area and shape, i.e., they are not necessarily congruent. Thus, a necessary condition that two triangles be congruent is that three elements of one be equal respectively to three corresponding elements of the other.

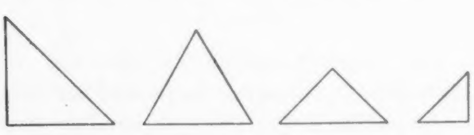

FIG. 1

Two triangles having three elements of one respectively equal to three corresponding elements of the other must have one of the following sets of elements equal: (a) three angles (Plate 3, top)—the triangles are similar (by definition); (b) two angles and the side opposite one of them (Plate 3, upper right and lower left)—the triangles are congruent; (c) two angles and the included side—the triangles are congruent; (d) two sides and the included angle—the triangles are congruent; (e) two sides and the angle opposite one of them (Figure 2)—two possible positions of one side for only one of which the triangles are congruent; (f) three sides—the triangles are congruent.

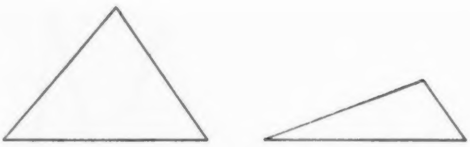

FIG. 2

The question then arises as to why in (b) and (c) it apparently made no difference whether the side was included between the two angles or not, whereas in (d) and (e) it was necessary to know what angle was involved. This presents an opportunity to introduce the fact that if two angles of one triangle are respectively equal to two angles of another triangle, then the third angles are equal also since the sum of the angles of a triangle is always 180°. This is easily demonstrated (Figure 3) by starting with the linkage on the right of Plate 1 at a zero angle and rotating the terminal side counterclockwise successively through the angles of any triangle—thus obtaining a straight angle.

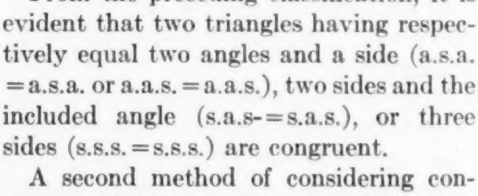

FIG. 3

From the preceding classification, it is evident that two triangles having respectively equal two angles and a side (a.s.a. =a.s.a. or a.a.s. =a.a.s.), two sides and the included angle (s.a.s-=s.a.s.), or three sides (s.s.s. =s.s.s.) are congruent.

A second method of considering congruent triangles makes use of linkages such as those in Figures 4A, B, C & D where the linkage in 4A is considered to be the original triangle and the others are used by the pupil to show that if any one of the conditions a.s.a. =a.s.a., a.a.s. =a.a.s., etc. is satisfied, then the triangles are congruent. Linkage 4B has a side equal to a

side of linkage 4A and the angles adjacent to that side may be set by reference to 4A so as to demonstrate a.s.a. = a.s.a.; 4B is also used to demonstrate a.a.s. = a.a.s.; 4C, s.a.s. = s.a.s.; and 4D, s.s.s. = s.s.s. Practically, linkages 4B, C, and D could be

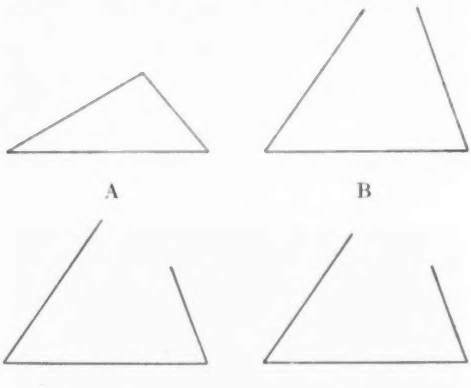

FIG. 4

combined into a single linkage, but that might be confusing to the pupils. In the case of a triangle having two sides and the angle opposite one of them equal to the corresponding parts of another triangle, it is very effective to use a linkage as in Figure 5A where two bars are used to represent one side of the original triangle. Then the two possible positions may be demonstrated simultaneously (Figure 5B).

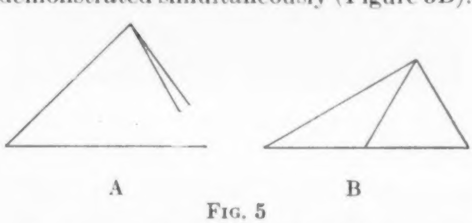

FIG. 5

Just as in the case of triangles, several types of quadrilaterals may be illustrated by means of linkages. In the case of quadrilaterals, however, linkages have an additional value in that they may be deformed. For example, the linkage at the top of Plate 4 may be used as a parallelogram, rectangle, or contra-parallelogram. By gradually changing the altitude of this figure it is possible to show very convincingly that the area of a parallelogram depends upon its altitude rather than its

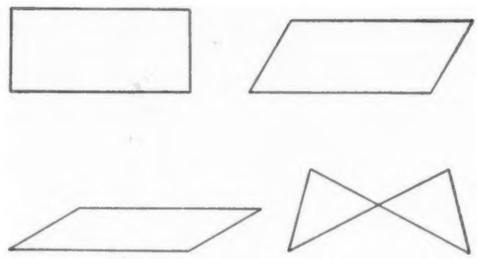

FIG. 6

width (Figure 6). Experience has shown that this demonstration has real value in assisting the slow pupil to master that concept. By making one diagonal of this linkage as long as possible and then pushing the two vertices at the center in the same direction, one obtains a contra-parallelogram. This shows that two pairs of equal opposite sides are not sufficient to determine a parallelogram in the ordinary form if we consider concave figures.

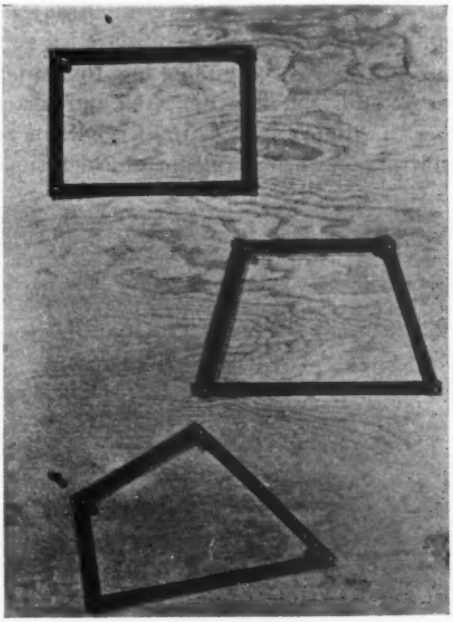

PLATE 4

The linkage in the center of Plate 4 represents an isosceles trapezoid and may be deformed to represent an arbitrary quadrilateral with one pair of opposite sides equal. The linkage at the bottom of

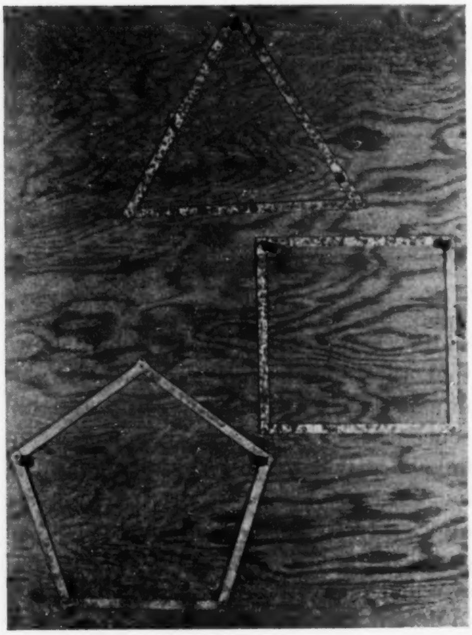

PLATE 5

Plate 4 represents a general quadrilateral and may be deformed to represent a general trapezoid (by unhooking the upper vertex and fastening the left hand vertex over the thumb tack at the left). Thus an arbitrary quadrilateral may be deformed

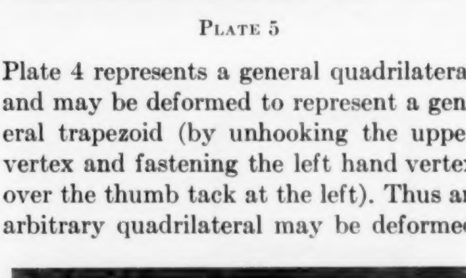

PLATE 6

into a trapezoid, i.e., a trapezoid is a special case of the position but not of the sides of a quadrilateral.

The regular polygons may be illustrated by linkages as in Plates 5, 6 and 7 where we have regular polygons of 3, 4, 5, 6, 8 and 10 sides inscribed in equal circles. These are useful in the lower grades in showing what the figures are like and illustrating the dependence, except in the case of the triangle, of regular polygons on both sides and angles. They are easily set in the

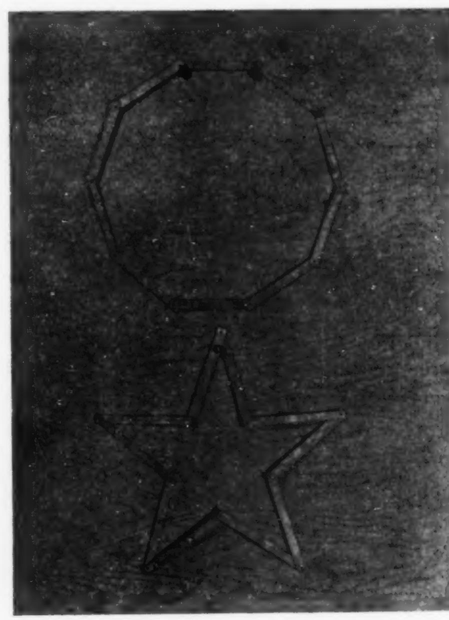

PLATE 7

proper position by comparing them with a full scale drawing inscribed in a circle. The necessity of having equal angles is very well shown in Plate 7 where a regular decagon has been deformed into a star. Also these linkages show that as the number of sides is increased the figure approaches that of a circle.

Before considering a few more complicated linkages, here is an original device (Plate 8) which has proven very useful when discussing two lines cut by a transversal. The transversal can be made to slant to the right or to the left or to be vertical, and thus the pupil does not come

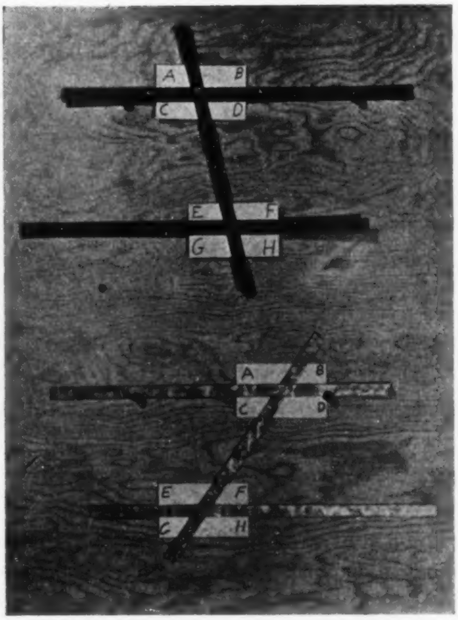

PLATE 8

to recognize it only when it is in a certain position. The cards are removable so that any method of notation for the angles may be used and changed easily. Since the cards are lettered on both sides, the instructor does not have to stand in front of the linkage. When metal strips are used, the joints can be made tight enough so that the bars stay where they are set regardless of how they are supported. Due to the easy deformability of the linkage it is possible to show the inaccuracy of any false statements made by the pupils and to bring out any principle desired.

We have considered several linkages which may be used to help the pupil visualize and understand the concepts he is taught. Next we shall consider a few linkages which are useful in arousing the pupil's interest in mathematics. They are sufficiently novel to appeal to him, and they illustrate mathematical principles which he may not understand but which give him some concept of the material he will study in more advanced courses.

LINKAGES TO AROUSE INTEREST

Most pupils are interested in the construction of a straight line or the trisection of angles; many have heard of conic sections, parallel rulers, and pantographs; and practically all are interested in mechanical devices which can be manipulated and made to perform various operations. We shall mention briefly some of the linkages which draw mathematically straight lines, draw conic sections, and trisect angles.

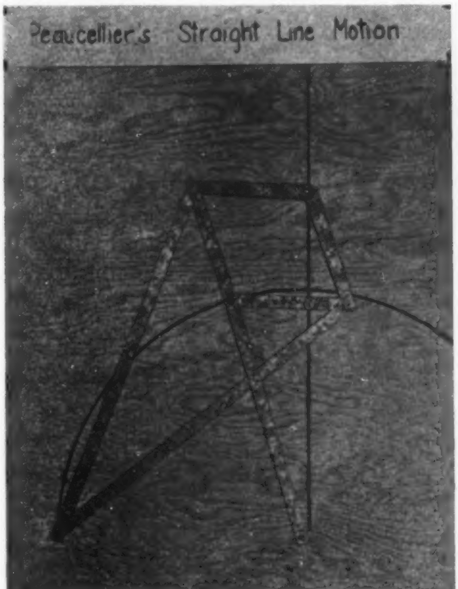

PLATE 9

There are two constructions which are fundamental to all Euclidean plane geometry—a circle and a straight line. The former is very simple and has previously been discussed; the latter has been assumed for thousands of years but was not proved until 1864. At that time an officer in the French army, A. Peaucellier, devised a linkage (Plate 9) making use of the fact that the inverse of a circle with respect to a point on it is a straight line. The fundamental element of this linkage is a rhombus, say $ABCD$ (Figure 7), two of whose opposite vertices, A and C, are joined by equal bars to a point O. It is called Peaucellier's Cell. This cell has the important property that the product of the distances OB and OD is a constant for any

given cell regardless of its position. This is easily proved algebraically as was shown

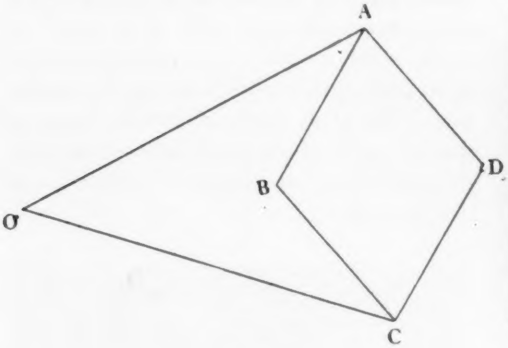

Fig. 7

by Joseph Hilsenrath in The Mathematics Teacher of October 1937, page 281.

Hart's Straight Line Motion (Plate 10) makes use of the same principles of inversion as Peaucellier's Straight Line Motion but replaces the cell by a contra-parallelogram each side of which is divided in the same ratio.[2]

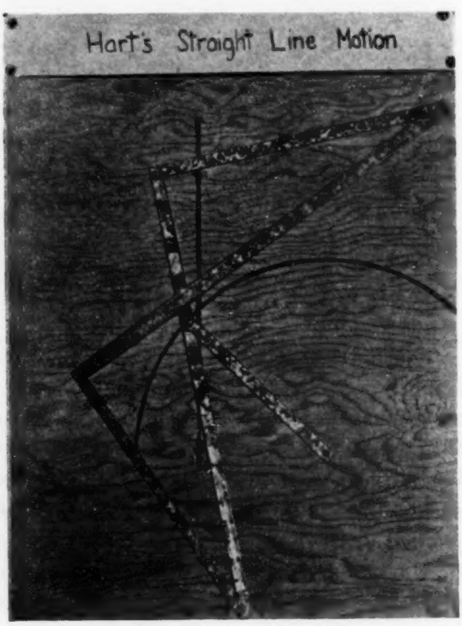

Plate 10

[2] Wm. D. Marks, "Peaucellier's Compound Compass and other Linkages," *Journal of the Franklin Institute*, CVII (June, 1879), p. 372.

Robert C. Yates in his book "Tools"[3] has pointed out that the inverse figures of the conics (Plate 11) may be obtained by using a contra-parallelogram with one side fixed and with points fixed on the opposite side at distances less than, equal to, and greater than half an adjacent side from one of its vertices. The figures traced are respectively the inverses of a hyperbola, parabola, and ellipse. Either the Peaucellier Cell as in Plate 9 or the contra-paral-

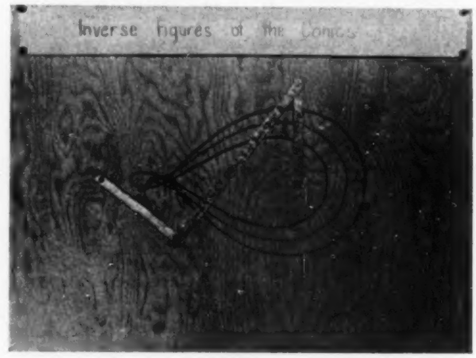

Plate 11

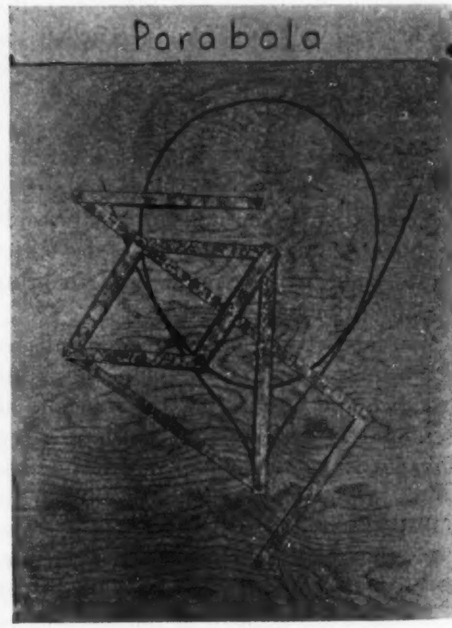

Plate 12

[3] Robert C. Yates, *Tools, A Mathematical Sketch and Model Book*. Louisiana State University, 1941.

lelogram as in Plate 10 may be used to obtain the conics from these inverse figures. In Plate 12 the Peaucellier Cell is used to obtain a parabola from its inverse.

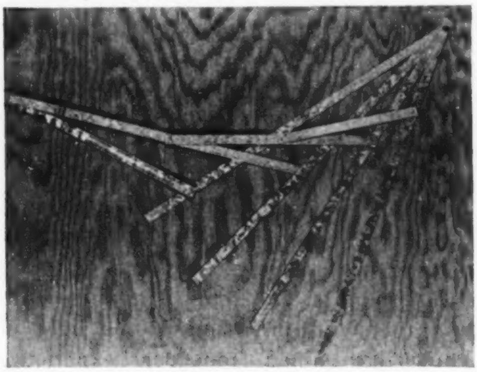

PLATE 13

The last linkage which we shall consider has always proved interesting to my pupils because they have heard of the problem. Many unsuccessful attempts (since it is impossible) have been made to trisect any angle by means of ruler and compasses. However, the trisection of any angle by means of linkages is not difficult. One of the best known angle trisectors (Plate 13) was devised in 1875 by Alfred Bray Kempe, a London barrister. It is based upon similar contra-parallelograms.[4]

[4] Joseph Hilsenrath, "Linkages," THE MATHEMATICS TEACHER, XXX (October, 1937), pp. 282–283.

There are also many other types of linkages such as the parallel rulers and the pantograph which are used in business today.

CONCLUSION

The use of linkages in teaching, except incidentally, seems to be rare. For several years I have used a set of linkages such as those in Plates 1–9 and 13 along with others for showing the congruence of triangles. One year I placed them on the molding above the blackboard across the front of the classroom and found them very useful in teaching the pupils to identify the various types as well as for reference so that I would not be constantly drawing right triangles, or some other figure, on the blackboard. They are also frequently useful in helping the pupil interpret worded problems.

In conclusion, it seems to me that we teachers may profit greatly by making, using and having the pupil use linkages as teaching devices, as visual aids. They represent something concrete rather than abstract "booklearning." They give to the pupil another concept of the dynamic nature of the world in which he lives rather than presenting static figures which he often does not translate into his everyday existence.

Triangulation According to James Fenimore Cooper in "The Prairie"

"The pursuit of bee-hunter is not uncommon on the skirts of American society. . . . When the bees are seen sucking the flowers, their pursuer contrives to capture one or two. He then chooses a proper spot, and suffering one to escape, the insect invariably takes its flight towards the hive. Changing his ground to a greater or a less distance, according to circumstances, the bee-hunter then permits another to escape. Having watched the courses of the bees, which is technically called lining, he is enabled to calculate the intersecting angle of the two lines, which is the hive."—Contributed by John H. Treanor, Francis Parkman School, Jamaica Plain, Mass.

A Program of Diagnostic Testing and Remedial Treatment in Arithmetic

By Joseph E. Fuller
Tuskegee Institute, Tuskegee, Alabama

I. The Need of Educational Diagnosis

It is common knowledge that a great many pupils find arithmetic to be difficult, and it is also generally known that the school population, as well as the general population, performs arithmetically on a level far below its normal expectancy. The results of tests by the Armed Forces involving arithmetic abilities merely tended to emphasize this fact.

As one's knowledge of how most effectively to guide learning is increased, as the materials of instruction become more nearly perfected, and as one is able to control the influences of the various elements of the environment—in proportion as one is able to do these things will the need for remedial instruction decrease. While much progress has been made along the lines, the need for remedial instruction is unavoidable at present, and the need will probably be acute for a long time to come.

According to F. P. Frutchey[1] "teaching may be based upon valid evidence carefully collected and wisely interpreted, or it may rest upon a series of untested assumptions, poor guesses, and wishful thinking—or some degree between the two."

There can be little, if any, doubt as to the proper basis of teaching. However, in actual practice one all too often pursues the method based upon untested assumptions, mere guesses, and wishful thinking.

II. The Meaning of Diagnosis

L. J. Brueckner states[2] that "educational diagnosis relates to the techniques by which one discovers and evaluates both strengths and weaknesses of the individual as a basis for more effective guidance. Diagnosis is a logical process based on a consideration of the available data concerning a particular individual or group of individuals. The analysis of these data and their interpretation in the light of knowledge gained from past experience enables the diagnostician to suggest necessary developmental or remedial measures."

III. The Techniques of Diagnosis

Any kind of learning can be diagnosed, whether or not that learning is important or desirable. If diagnosis is to occupy its rightful place in effective teaching, it is necessary that diagnosis be made in connection with worthy objectives of the educational program. It is not my purpose here to discuss the objectives of arithmetic. It is sufficient for this discussion to say that in making a diagnosis, all of the objectives should be kept in mind. Diagnosis in arithmetic must not only concern itself with difficulties in computation, but also with difficulties in the other objectives of reasoning and problem solving.

The first steps, therefore, in diagnosis is to determine the objectives of arithmetic.

From the definition of diagnosis it is seen that diagnosis involves two steps. The first is measurement, and the second is interpretation. One of the tools of measurement is the general intelligence test. Brueckner states[3] that "if one accepts the definition of intelligence as the ability to learn, and assumes that these tests measure this aspect of mentality, one can readily determine the relative ability of an individual."

[1] Frutchey, F. P., "Educational Prevention, Diagnosis and Remediation." *Review of Education Research*, Vol. 8 (December, 1938), 513.

[2] Brueckner, L. J., "Need of Educational Diagnosis." *National Society for the Study of Education, Thirty-fourth Yearbook*, 2.

[3] Brueckner, L. J., *op. cit.*, 9.

Too much importance must not be attached to the intelligence test. A low score on these tests probably indicates that at least one factor behind his inability to learn may be due to the lack of innate ability. Intelligence test results at best give a guide for further observation.

In addition to a general intelligence test standard achievement tests in arithmetic should be given. Brueckner states[4] that "these tests should be of two types. 1. Those which test general ability in arithmetic. Such tests have as their purpose the measurement of general traits, such as computational ability, knowledge of vocabulary, ability to apply quantitative techniques in the solution of problems. General survey tests enable the teacher to determine the general phases of instruction that have been adequately stressed, and those that must be more fully developed; such tests also enable the teacher to select those pupils whose performances are below standard. 2. Those which test specific abilities in arithmetic. These tests which are analytical in character are given to those pupils whose performances are below standard." Such tests are designed to locate specific weaknesses, both in number processes and in problem solving.

In as much as reading ability is fundamental to success in other subjects, a test of reading ability should be given.

Stenquist[5] lists the following criteria in choosing diagnostic test materials:

1. Will the use of the test tend to center attention upon a desirable aspect of the subject?
2. Does the test isolate the specific elements in the total learning, so that a true diagnosis may be made?
3. Is the test a valid measure of what it purports to measure?
4. Does the test afford sufficient repetitions of a specific item, so as to give a genuine diagnosis?
5. Does the material utilize all relevant motives for learning the associations being fixed?
6. As a check on levels of attainments, does the test have recent, well-defined, and comparable norms?
7. Is the diagnostic material sufficiently reasonable in price to permit its use?

In any program of diagnostic and remedial instruction the keeping of records is imperative. The more complete the record of each pupil, other things being equal, the more reliable can be the diagnosis. Since many of the records must be kept by the teacher, the extent of the records kept must vary. However, it seems that the minimum data should be the following: (1) the I.Q. for each pupil, (2) the median I.Q. for each class, (3) the I.Q. zones, (4) grade level of achievement for each pupil, (5) profile chart for each pupil showing, achievement in relation to age, I.Q. and grade attained, and achievement in relation to I.Q.

Some form of cumulative permanent record for each pupil is of particular importance for his educational guidance.

In the interpretation of results additional information such as personal data sheets, physical examination data should be used. Certain subjective elements such as the teacher's observation of the pupil at work both in his oral and written assignments, interviews with the pupil, and others who may have information concerning the pupil are valuable aids in interpreting the results. Estimates of emotional reactions, persistence, self-confidence, attention to details, conscientiousness, sociability, and general emotional stability should also be considered.

The task of interpreting consists of drawing inferences and conclusions. How valid these inferences and conclusions are is determined by the experience of the diagnostician, and by the carefulness of his thinking.

In arithmetic where the measurement deals directly with desirable or undesirable educational outcomes, the results can be

[4] Brueckner, L. J., "Diagnosis in Arithmetic," *National Society for the Study of Education, Thirty-fourth Yearbook*, 283.

[5] Stenquist, "The Administration of a Program of Diagnosis and Remedial Instruction," *National Society for the Study of Education, Thirty-fourth Yearbook*, 512–513.

directly interpreted as indicating those pupils who are attaining or failing to attain these desirable educational objectives.

According to R. W. Tyler[6] "a satisfactory diagnosis must, (1) concern itself with worthwhile objectives, (2) provide valid evidence of strengths and weaknesses related to the objectives, (3) be reasonably objective, so that other competent persons may arrive at similar conclusions in following the same diagnostic procedure, (4) be reliable, so that additional diagnoses covering other samples of pupil reactions will not give widely different results, (5) be carried to a satisfactory level of specificity, (6) provide comparable data, (7) be comprehensive, (8) be appropriate to the program of education desired, (9) be practical, and (10) should be conducted by persons who are well qualified as educational diagnosticians. As these characteristics are increasingly well met, we may expect educational diagnoses to yield more fruitful results."

IV. The Remedial Program

The testing, the making of charts, the keeping of records, and every tool used as an aid in diagnosis are merely preliminary to the remedial program. There is no need to locate weaknesses unless something is done about them.

The first question which naturally arises is, who shall take remedial work? A pupil is a candidate for a class in remedial instruction when his achievement is obviously at a lower level than that which the indexes of his ability demonstrate he is capable of attaining. Hanson[7] says, "if a student's achievement is on the level of his abilities, then remedial classes are not for that individual."

What should be done with the pupil whose achievement is on the level of his ability, and who falls below the arbitrary standard? For such a pupil a program should be offered in keeping with his ability. This applies to the bright pupil as well as to the dull pupil. Such a program would not then be remedial, but a normal program for such a pupil. In the mechanical field it is not expected that one will get the same power from a ten horsepower engine as from a twenty horsepower engine, but in school work one, too often, expects and even demands achievement beyond the pupil's ability.

Remedial work then should be required of pupils whose achievement is below the level of their ability. It may be elective for others who may desire it.

Remedial classes should be scheduled as any other class is scheduled, and may or may not yield credit. The name of the class should be of such a nature to cause as little embarrassment as possible to those required to take remedial work.

W. J. Reilly states,[8] "it is the removal of the student from the classroom situation to a more private situation in which the teacher may more closely observe and study the reasons for the student's lack of proper responses to projected stimuli. Remedial teaching should necessarily be an individual-teacher relationship."

The method of instruction should be individualized work within the group, so as to make provision for the wide range of individual achievement and the differences in learning abilities. Each pupil may have different difficulties, and may need varying amounts of practice.

The materials used should be such that much self-instruction is possible, and they should be highly flexible. There should be self-administering tests so that the pupil can check his own progress.

Diagnostic tests in arithmetic reveal that individual difficulties fall in one or more of the following classes: (1) cases in which difficulty is due to lack of skill in

[6] Tyler, R. W., "Characteristics of a Satisfactory Diagnosis," *National Society for the Study of Education, Thirty-fourth Yearbook*, 110–111.

[7] Hanson, E. H., "Remedial Instruction in Rock Island," *American School Board Journal*, (March, 1942), 50.

[8] Reilly, W. J., "Phases of Remedial Teaching in the Bradford City Schools," *Pennsylvania School Journal* (June, 1938), 327.

fundamentals, (2) cases of difficulties with complex situations involving fundamental operations, and (3) cases of difficulty in problem solving.

From the results of diagnostic achievement tests the errors or types of errors made should be tabulated for each student. The pupil is then given drill work on these difficulties, one class of difficulty at a time, until mastered. Tests are given at intervals until few, if any, errors are made on successive tests, and the time required for successive tests of the same weakness is practically the same.

In remedial work the pupil should start with concrete situations which he is able to comprehend, otherwise arithmetic may seem to him a meaningless set of symbols and words. In solving the concrete problems he becomes interested in the principles involved. He goes on to other problems involving the same or similar principles. In this way he begins to understand the operations by which certain types of problems can be solved. He begins to formulate rules of his own. He is, therefore, able to understand the rules formulated by others and to use them intelligently in the solution of problems.

"Those who have made analyses of errors and methods of work have clarified many issues relative to the characteristics of learning. For some difficulties suitable remedial exercises have been suggested; yet little exact information is available concerning the effectiveness of the various proposed remedial measures for correcting particular kinds of difficulties. That many difficulties can easily be corrected is apparent from the results of ordinary instruction. Teachers, individually and as a group, have accumulated a mass of techniques that they apply with varying degrees of assurance and success."[9]

Whatever the method used the pupil must work out things for himself. In so doing he not only develops concepts, but builds up techniques of attacking problems. The teacher must remain in the background, giving the pupil the benefit of his methods only when needed.

In any learning the mental attitude of the learner is an important factor. Therefore, real motivation should be a feature of all class work. This can be partially achieved by emphasizing the practical applications of arithmetic. The pupil should always be aware of the progress he is making. To this end the pupil should keep progress charts and graphs so that he can observe his progress on practice exercises as well as his general growth as determined by standardized tests. Brueckner[10] says, "classes in which pupils have definite knowledge of their progress make considerably more improvement than classes without such knowledge."

Thompson[11] states "that remedial instruction should be scheduled for at least a term, and the number of terms will be determined by the extent of the student's retardation." However, more than a year's work is not recommended.

The number of periods per week should be in keeping with the number of periods for other classes.

The teachers of remedial classes should be trained in remedial methods as well as in subject matter methods. When it is not possible to have teachers specifically trained in remedial methods, teachers can improve themselves greatly for remedial instruction by the study of professional books on remedial methods.

Progress should be measured in terms of the pupil's improvement as compared with his previous record, and not necessarily with any arbitrary standards of achievement. He is competing primarily with himself.

An important problem in any diagnostic and remedial program is that of evalua-

[9] Brueckner, L. J., "Technique of Diagnosis," *National Society for the Study of Education, Thirty-fourth Yearbook*, 153.

[10] Brueckner, L. J., "Diagnosis in Arithmetic," *National Society for the Study of Education, Thirty-fourth Yearbook*, 298.

[11] Thompson, R. B., "Administration of Remedial Programs," *Educational Administration and Supervision*, Vol. 27 (March, 1941), 226.

tion. The standards to use in evaluating the reactions of pupils are objectives which have been set up as the goals. As a step in the process the results of tests given at intervals during the remedial period are tabulated and studied. At the end of the first remedial period the pupil is retested on the original achievement test, or another form, of tested reliability, of the original achievement test. The results should be compared, and translated in terms of grade equivalents. A new profile chart for each pupil showing achievement in relation to his I.Q. should be made.

Some of the difficulties in evaluation are those of obtaining greater objectivity, and of obtaining measurements in fine enough units for purposes of exact appraisal. Even in the objective tests personal judgment in determining the degree to which the possible responses represent the attainment or non-attainment of desired objectives must be a factor. However, standard tests represent to date our most accurate method of evaluation, and educators are constantly striving to improve tests both as to objectivity and in making tests to measure smaller and smaller units. In as much as remedial instruction should be on an individual basis, the teacher has a real opportunity to know each pupil. This subjective knowledge should be used in the final interpretation of results.

V. Conclusion

Any program of diagnostic and remedial instruction to be effective must be practicable. Is it possible for the particular school to put the program into effect, that is, does the program require more time, personnel and equipment than can be provided is an important question to be answered before attempting a program of diagnostic and remedial instruction.

"Diagnosis should be a continuous process carried on through the entire learning activity. Diagnosis should not be limited to the beginning stages of work and the testing and analysis that are done after the learning of some skill has been completed. Periodic and systematic studies of the activities of the learner should be undertaken regularly while he is progressing in the learning of some ability or skill so that we can discover at as early a time as possible any conditions likely to interfere with successful mastery or evidences that learning is not proceeding smoothly and happily."[12]

[12] Brueckner, L. J., "Necessity of a Continuous Program of Diagnosis." *Journal of Educational Research*, Vol. 35 (February, 1942), 459.

THE MATHEMATICS TEACHER TAKES THIS OPPORTUNITY TO WISH ALL OF ITS READERS AND SUBSCRIBERS A VERY MERRY CHRISTMAS AND A HAPPY NEW YEAR!

An Attempt to Eliminate Common Errors in Elementary Algebra

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One of the most discouraging problems which confronts the teacher of both elementary and intermediate algebra is the everyday recurrence of fundamental errors in handling algebraic fractions and solving equations, namely, destruction of the values of the fractions by faulty cancellation, and destruction of equality. The present paper is a summary of an effort which the author has used in an attempt to reduce the number of such errors, and which he has found to be more or less successful.

When analyzed, all errors of faulty cancellation or destruction of equality can be reduced to the student's inability to realize when he is dealing with a sum of terms or with an indicated product of factors, and his attendant lack of understanding of how to perform the operations of addition, subtraction, multiplication, and division on either type of expression. To substantiate this conclusion several sets of arithmetics used in grade schools were consulted and found to contain no information on this topic. Frequent and sufficient repetition of the primary principle for handling fractions in these texts eliminates the possibility that this is the source of the errors in using fractions. These arithmetics also taught the principle of preservation of equality without making specific distinction as to whether either or both sides of an equation were factored or unfactored expressions. In other words students are not taught in grade school the operation upon a factored expression as distinguished from operation upon an unfactored one. This may be justified in arithmetic since there is not an immediate or necessary reason for teaching it. However in algebra there is a definite reason for the student being made acutely aware of the type of expression he is working with and how to operate upon it. Hence it is recommended that teachers in the elementary grades teach this distinction as a preparatory background for elementary algebra.

The following set of notes and illustrations (designed to be placed in the hands of the students at the outset of the course in elementary algebra), is destined to try to make up for this omission. It should be recognized that many of the facts or statements, called "rules," in these notes may appear to be absurdly self-evident, but it is exactly because of their obviousness, that students fail to be conscious of them and repeatedly violate them. Following the notes are a few remarks as to how the students are referred back to these rules in explaining why they should not add unlikes, and why some of their errors in faulty cancellation or destruction of equality are nothing more than violations of these rules.

Notes for Student Use

Definitions

A factor of a given number is one of two or more numbers whose product is the given number.

A term may be either a single number or letter, or any group of factors (i.e., any indicated product).

A monomial is an expression of only one term.

A binomial is the expression of the sum of two terms.

A trinomial is the expression of the sum of three terms.

A polynomial is the expression of the sum of three or more terms.

(Since a term consists of a group of factors, a factor might be considered to be a "part of a term.")

A factored expression is a single termed expression. (The factors may be poly-

nomials, yet the expression as a whole is said to be a monomial.) It will be recognized by the fact that when viewed as a whole, a product of factors is indicated.

An unfactored expression is an indicated sum of two or more terms. (In this sense a term might be considered to be a "part of an unfactored expression." Any factor may itself be an unfactored expression.)

KEEP IN MIND: Terms are ALWAYS separated from one another by a positive or negative sign. Factors are NEVER separated from one another by a positive or negative sign.

Ways of Indicating a Product of Factors

1) Use of the "times" sign, $\times$: $A\times B$

2) Use of the dot: $A\cdot B$

3) Use of parentheses: $(A)(B)$; $(A-B)(C-D)$

4) No sign between letters: ab (This method is limited to the indication of products of single letters; it is not used in arithmetic, i.e., its use is exclusive to algebra.)

Recognition of Factored and Unfactored Expressions

Factored Expressions: (Monomials, Single Terms)

1) t

2) st

3) $C(E-D)$

4) $(m-n-t)(c)(A-B)$

5) $(A-B)(C-D)$

Unfactored Expressions: (Bi-, Tri-, or Polynomials)

1) $s-t$

2) $ab-bc-cd$

3) $A+Bt$

4) $(a+b)(x-y)-(a-b)(x+y)$

5) $c(a-b)+d$

6) $s+\frac{1}{2}(c-d)(m-n)$

Throughout the study of algebra we will be constantly forced to consider the questions: 1) Is this expression a factored or an unfactored one?, and 2) Can I transform this unfactored expression to a factored one? And having answered these questions we must know how to operate on either kind of expression.

Operations Upon Unfactored Expressions

Multiplication and Division Applied to Unfactored Expressions:

Given the following simple problems in computation:

1) $5(2+3+4)=?$

and

2) $\dfrac{3+6+9}{3}=?$

From our knowledge of the operations indicated by these symbols, we know that we must multiply the sum of 2, 3, and 4 by 5, i.e., multiply 9 by 5, and divide the sum of 3, 6, and 9 by 3, i.e., divide 18 by 3. Thus a correct solution to both problems would be:

1) $5(2+3+4)=5(9)=45$

2) $\dfrac{3+6+9}{3}=\dfrac{18}{3}=6$

Now it so happens that there is another way of obtaining the correct results; this, second method, which is unnecessary in arithmetic, is absolutely essential in the study of algebra, because it will not always be possible to obtain the sum indicated within the parentheses. Here we will merely illustrate this method and thereby show that it leads to the correct results, and also illustrate with the same set of numbers a procedure that beginning students in algebra attempt to employ but which leads to an incorrect solution.

Correct Procedure or Alternate Method for Multiplying or Dividing a Sum of Terms by Another Number

$$5(2+3+4)=45$$

$$(5)(2)+(5)(3)+(5)(4)\overset{?}{=}45$$

$$10 + 15 + 20 \overset{?}{=} 45$$

$$45 = 45$$

$$\frac{3+6+9}{3}=6$$

$$\frac{3}{3}+\frac{6}{3}+\frac{9}{3}\overset{?}{=}6$$

$$1+2+3\overset{?}{=}6$$

$$6=6$$

An Incorrect Procedure Which Students Are Generally Tempted to Employ

$$5(2+3+4)=45$$

$$(5)(2)+3+4\overset{?}{=}45$$

$$10+3+4\overset{?}{=}45$$

$$17\neq 45$$

$$\frac{3+6+9}{3}=6$$

$$\frac{3+\overset{2}{\cancel{6}}+9}{\cancel{3}}\overset{?}{=}6$$

$$3+2+9\overset{?}{=}6$$

$$14\neq 6$$

From these illustrations we may formulate the following rule:

To multiply or divide an unfactored expression (i.e. a sum of terms) by any number or letter, EVERY TERM of the expression must be multiplied or divided by that number or letter.

Stating this rule in terms of algebraic symbolism we have:

$$a(b+c+d)=ab+ac+ad$$

$$\frac{b+c+d}{a}=\frac{b}{a}+\frac{c}{a}+\frac{d}{a}.$$

(This is usually referred to as the Distributive Law, i.e., the effect of a multiplier or divisor is felt by or distributed to all the terms of a sum.)

Addition and Subtraction Applied to Unfactored Expressions:

Given the following simple problem in computation:

a) $(2+3+4)+5=?$

b) $(6+5+4)-2=?$

From an elementary point of view the solution to either of these problems seems extraordinarily obvious.

a) $(2+3+4)+5=9+5=14$

b) $(6+5+4)-2=15-2=13$

But we might have also obtained the correct answers by adding the 5 or subtracting 2, to or from any one of the terms in the respective problems, and added the results to the sum of the other two terms, as follows:

$2+3+4=\ 9$	$2+3+4=\ 9$	$2+3+4=\ 9$
$5=\ 5$	$5\ =\ 5$	$5\ =\ 5$
$2+3+9\overset{?}{=}\ 14$	$2+8+4\overset{?}{=}\ 14$	$7+3+4\overset{?}{=}\ 14$
$14=\ 14$	$14=\ 14$	$14=\ 14$
$6+5+4=\ 15$	$6+5+4=\ 15$	$6+5+4=\ 15$
$-2=-2$	$-2\ =-2$	$-2\ =-2$
$6+5+2\overset{?}{=}\ 13$	$6+3+4\overset{?}{=}\ 13$	$4+5+4\overset{?}{=}\ 13$
$13=\ 13$	$13=\ 13$	$13=\ 13$

By way of foreseeing an error which you may be tempted to make or will make in the future in handling algebraic expressions, the following illustrations should prove that addition or subtraction to every term of a sum does not add or subtract that number to or from the sum.

$$\begin{array}{r} 2+3+4=\ 9 \\ +5+5+5 \quad 5 \\ \hline 7+8+9\overset{?}{=}14 \\ 24\neq 14 \end{array} \qquad \begin{array}{r} 6+5+4=\ 15 \\ -2-2-2 \quad -2 \\ \hline 4+3+2\overset{?}{=}13 \\ 9\neq 13 \end{array}$$

a) $(2)(4)(8)\times 2=?$ b) $\dfrac{(2)(4)(8)}{2}=?$

Again the results are quite obvious:

$$(2)(4)(8)\times 2=(64)(2)=128$$

$$\frac{(2)(4)(8)}{2}=\frac{64}{2}=32$$

However we might also have obtained the correct results if we multiplied or divided any ONE of the factors by the multiplier or divisor and taken the resulting product or quotient. For example:

$$\begin{array}{r} (2)(4)(8)\ =\ 64 \\ 3\ =\ 3 \\ \hline (2)(4)(24)\overset{?}{=}192 \\ 192\equiv 192 \\ \dfrac{(2)(4)(8)}{2}=\dfrac{64}{2} \\ (1)(4)(8)\ \overset{?}{=}\ 32 \\ 32\equiv\ 32 \end{array} \qquad \begin{array}{r} (2)(4)(8)\ =\ 64 \\ 3\ =\ 3 \\ \hline (2)(12)(8)\overset{?}{=}192 \\ 192\equiv 192 \\ \dfrac{(2)(4)(8)}{2}=\dfrac{64}{2} \\ (2)(2)(8)\ \overset{?}{=}\ 32 \\ 32\equiv\ 32 \end{array} \qquad \begin{array}{r} (2)(4)(8)\ =\ 64 \\ 3\ =\ 3 \\ \hline (6)(4)(8)\ \overset{?}{=}192 \\ 192\equiv 192 \\ \dfrac{(2)(4)(8)}{2}=\dfrac{64}{2} \\ (2)(4)(4)\ \overset{?}{=}\ 32 \\ 32\equiv\ 32 \end{array}$$

Therefore we may state the following fact as a rule: To add or subtract to or from an indicated sum of terms, add to or subtract from ONLY ONE term of the sum.

Stating this second rule in algebraic symbols we have:

To find:

$(a+b+c)\pm d=$ a sum of terms $\pm d$.

Solution:

$$a+(b\pm d)+c$$
$$(a\pm d)+b+c$$
$$a+b+(c\pm d)$$

Operations Upon Factored Expressions

Multiplication and Division Applied to Factored Expressions:

The problem in arithmetic might be either of the following:

An error which you are likely to fall into in handling algebraic expressions of this type is to multiply or divide every factor by the multiplier or divisor. The following illustration should prove to you that such a procedure will lead to an incorrect solution.

$$\begin{array}{r} (2)(4)(8)\times 2=128 \\ (4)(8)(16)\ \overset{?}{=}128 \\ 512\neq 128 \end{array} \qquad \begin{array}{r} \dfrac{(2)(4)(8)}{2}=32 \\ (1)(2)(4)\overset{?}{=}32 \\ 8\neq 32 \end{array}$$

Therefore we may state a third rule: To multiply or divide a factored expression by a given number of letter multiply or divide ONE of the factors by that number or letter, and obtain the product of the resulting factors.

Stating this rule in algebraic symbols we have:

$$[(a)(b)(c)]\times d=(ad)(b)(c)$$
$$=(a)(bd)(c)$$
$$=(a)(b)(cd)$$

$$\frac{(a)(b)(c)}{d}=\left(\frac{a}{d}\right)(b)(c)$$
$$=(a)\left(\frac{b}{d}\right)(c)$$
$$=(a)(b)\left(\frac{c}{d}\right)$$

Addition and Subtraction Applied to Factored Expressions:

Finally we take the problems in arithmetic:

$(3)(4)(5)+2=?$ and $(3)(4)(5)-2=?$

Concerning the order of operations we know that it is internationally agreed that the operation of multiplication takes precedence over addition or subtraction, Thus there is only one way in which we can compute the given problems: We must first find the product of the factors before adding or subtracting 2, as follows:

$$(3)(4)(5)+2=60+2 \qquad (3)(4)(5)-2=60-2$$
$$=62 \qquad\qquad =58$$

It can be seen that adding or subtracting to or from one of the factors or all of them does not produce the same results:

$$\begin{array}{rr} (3)(4)(5)= & 60 \\ +2 & +2 \\ \hline (3)(6)(5)\overset{?}{=} & 62 \\ 90\neq & 62 \end{array} \qquad \begin{array}{rr} (3)(4)(5)= & 60 \\ -2 & -2 \\ \hline (1)(4)(5)\overset{?}{=} & 58 \\ 20\neq & 58 \end{array}$$

Therefore we have a fourth and final rule: To add or subtract, to or from, a factored expression we add or subtract only after the product of factors has been obtained.

Restatement of this rule in algebraic symbols yields:

$$(a)(b)(c)\pm d=(abc)\pm d.$$

Conclusion

It is quite possible that many teachers after laboring for the first time through the set of notes just presented, will wonder just how they could be made to seem important to the students or even perhaps how they can be of use to them. The following concluding remarks has proved to the author that such dwelling on the obvious is not in vain and has a definite pedagogical value. Superior and average students especially will carry out many operations with letters much more meaningfully, when they are conscious of applying these laws.

The following are some of the common errors students will make. All of these errors can be explained as violations of these rules which we have stated.

In solving simple linear equations students destroy equality because they do not examine either side of the equation as to whether it is factored or unfactored, and as a result do not multiply the entire member of an equation by a number when they think they have. They feel that because they have performed some operation on both sides they have preserved equality, and mere lack of appreciation of how to go about performing the four fundamental operations on factored expressions as distinguished from unfactored ones is the real cause of their difficulty.

In simplifying fractions the value of the fraction is changed because in applying the fundamental principle regarding the handling of fractions they have not distinguished when they have actually divided the entire numerator or denominator by the same number or expression. Again the error fundamentally was a lack of consciousness as to whether or not the numerator or denominator was a factored or unfactored expression.

Lastly the stubborn persistence of the error of addition or subtraction of unlikes may be explained from a different point of

view. We may consider the addition of $3x+4$ to obtain the incorrect sum, $7x$, a violation of Rule 4, which states that we have no right to add to either factor of a factored expression; that the product must be obtained first before addition or subtraction is possible. This latter product cannot be obtained unless a specific value is assigned to x. As a consequence the addition of $3x$ to 4 must remain an indicated sum.

Of course it is taken for granted that the examples of how to go about operating on factored and unfactored expressions in arithmetic are not to be taken as proofs that the parallel procedures in algebra are the correct procedures, but are merely to be taken as illustrations that such procedures (rules) yield correct results. In a sense these rules and illustrations are to be taken as undefined principles or postulates upon which we build the entire "science of equations."

◆ THE ART OF TEACHING ◆

A Graphical Solution of Some Common Problems

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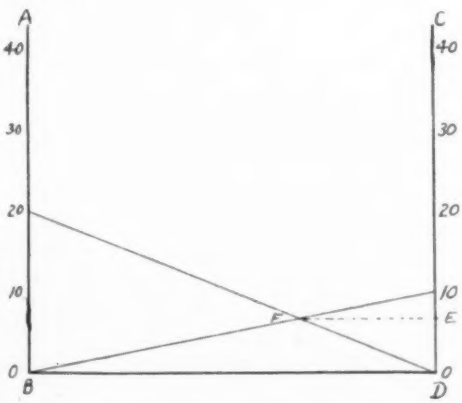
Several types of problems can be solved by the same kind of graphs.

Problem 1. If A can do a piece of work in 10 days and B can do the same work in 20 days, how long will it take both of them to do the work?



Solution. Draw BA and DC perpendicular to BD. On BA lay off the number of days B takes to do the work. On DC lay off the number of days A takes to do the work. Join each of these points to the foot of the other perpendicular. These lines intersect at F. The distance from F to BD is the time it will take both to do the work. If we know the time each of three men would require, find the time two of them could do it and then the time needed by the two and the third.

Problem 2. If two electric conductors are connected in parallel, $R_1=20$ ohms, $R_2=10$ ohms their combined resistance would be found similarly as in Prob. 1 to be $6\frac{2}{3}$ ohms.

Problem 3. If an airplane travels in a certain direction at the rate of 200 miles an hour and return at 100 miles an hour, how far can it go and return in one hour? Increase the scales on BA and DC to ten times the scale used and in the same way find that the plane could go $66\frac{2}{3}$ miles and return in one hour. If it carried fuel for 10 hours of flying it would have a cruising range of 667 miles, with no factor of safety.

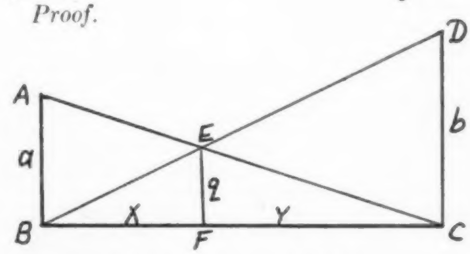
Problem 4. A formula used in physics for mirrors and lenses is

$$\frac{1}{f}=\frac{1}{u}+\frac{1}{v}.$$

This problem can be solved graphically by using the same procedure. Lay off $U=10$, the distance of the object on DC. $V=20$, the distance of the image on BA and read off $f=6\frac{2}{3}$ the focal distance of lens.

Problem 5. Find the quotient of the product of two numbers divided by their sum. If a and b are the numbers r is the quotient. q

Proof.



1. $\triangle ABC$ is similar to $\triangle EFC$
2. $\dfrac{q}{a}=\dfrac{y}{x+y}$
3. $\triangle BCD$ is similar to $\triangle BFE$
4. $\dfrac{q}{b}=\dfrac{x}{x+y}$
5. $\dfrac{q}{a}+\dfrac{q}{b}=\dfrac{x+y}{x+y}=1$
6. $bq+aq=ab$
7. $q=\dfrac{ab}{a+b}$ Q.E.D.

◆ IN OTHER PERIODICALS ◆

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The American Mathematical Monthly

August–September 1946, Vol. 53, No. 7.

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